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MATHEMATICAL CENTRE TRACTS 160

ABSTRACT AUTOMATH

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"De objectiveering der wereld in wiskundige systemen bij verschillende individuen wordt in onderling verband gehouden door de passielooze taal,die bij den hoorder het identieke wiskundige systeem als bij den spreker doet oprijzen,terwijl de gevoelsinhoud van dat systeem bij beiden totaal verschillend kan zijn..."

(L.E.J.BROUWER: "Wiskunde en Ervaring".)

P R E F A C E

The idea of a computer-assisted proof-checking (applied to concrete proofs, as they appear in the mathematician's everyday life) has probably occurred to many minds, even before the von Neumann computer was conceived.
In particular, an attempt to increase the reliability of mathematical texts (proofs) by having them processed and checked on computer has also been the main motivation behind the specific AUTOmated MATHematics Project, developed since 1967 at the Eindhoven University of Technology.

Abstracting from the pragmatic motivations (cf. 02. below), a sufficiently complex AUTOMATH-system (the "classical" version: AUT-68, say) can be viewed as an (applied) typed lambda-calculus with a "polymorphic" type-structure (to coin a word from R.Milner).
Although the underlying "type-polymorphism" appears to be - at a first look - a very specific one, we realize soon that an AUTOMATH-system is powerful enough such as to allow interpreting (in it) familiar typed lambda-calculi as, e.g., the Curry Theory of Functionality (= "First-order Typed Lambda-calculus") or the Girard-Reynolds Second-order ("parametric") Typed Lambda-calculus. In this respect, an AUTOMATH-system is a "generalized typed lambda-calculus".

At a closer examination, the "pure" (i.e., constantless) part of these systems turns out to be much similar if not equivalent to other typed lambda-calculi occurring in the recent literature on the foundations of logic and mathematics. (Specifically, "Pure AUT-68" is, in fact, a fully formalized version of J.P.Seldin's Theory of Generalized Functionality (cf. Annals Math. Log. 17, 1979, 29-59), and it becomes - under an appropriate translation - equivalent to the "Π-fragment" of Martin-Löf's Theori(es) of Types (cf. References, under MARTIN-LÖF.)

On the other hand, the specific manipulation of the AUT(OMATH)-constants - a *sui generis* feature of these systems - is, up to a certain point, independent from the underlying lambda-calculus part and can be used, with similar effects, in connection with any other typed lambda-calculus.

The present work is intended to discuss in detail a theoretical aspect of the main AUTOMATH-systems,viz. the possibility of "separating" the "lambda-calculus-free" part (usually identified as "Primitive AUTOMATH") from a "full AUT-system".Despite the fact that the so-called "Primitive AUTOMATH" can be defined independently,it is not immediately clear,from the language-definition of "bigger" systems, that the corresponding "correctness categories" of the latter are actually "conservative" over those of the former one.
The affirmative answer (given in 33.,below) insures the fact that the "definitional mechanism" of an AUT-system is actually an independent,"super-imposed" structure on a "Pure AUTOMATH"-system and shows that a (theoretical) study of the latter (which is,properly speaking, a Chapter of Lambda-calculus) might be also profitable for the specific purposes of the AUTOMATH Project.

Other theoretical aspects of these systems,as,e.g.,a "global proof-theory" and a "mathematical" semantics (model theory) will be discussed elsewhere.

ACKNOWLEDGEMENTS.

This book is the outcome of a not too remote and particularly fruitful period of time when I was employed by the Eindhoven University of Technology ("THE"),in the Department of Mathematics and Computing Science.

It is probably premature,for me,to be aware of the various influences I have undergone during my stay in Eindhoven.Still,I have a number of specific debts to record.

First among these is to Professor N.G.de Bruijn who has suggested the subject of this work and has been the main source of counsel,encouragement and criticism during the months I was writing the book.
His views on AUTOMATH and (Typed) Lambda-calculus,in general,have exerced a decisive influence on the present work.

Next I am indebted to my colleagues in the AUTOMATH research group, who have helped me improve my understanding of the subject-matter.

I would like to thank,in particular,L.S.van Benthem Jutting,who has spent - altogether - long weeks on several "preliminary" drafts of the main text,suggesting numerous improvements and providing detailed criticism.

Many ideas on a rigorous language description of the main AUTOMATH systems - which have been used in the book - should be actually credited to D.T.van Daalen,who advised me on a couple of critical aspects of the proof-theory of AUTOMATH.

Of course,I still remain the only responsible one for the possible shortcomings of the text.

My debts to the literature should be obvious from the long list of References,placed at the end of the book.(Quotations are given by author and year of publication/pre-print - the last two digits - for all items that are not official AUT-reports,while the AUT-reports are referred to by author,year of publication/pre-print and report number.So DE BRUIJN 80-72 is Number 72 in the official THE-list of the AUT-reports.)

At this point I should also acknowledge,again,my debt to Henk Barendregt who taught me Lambda-calculus.

Hartfelt thanks to a philosopher: David Meredith,for reading and commenting on an early version of the Introduction.(I should perhaps ultimately agree with his "Proof is talk." by humbly noticing that it is so once we have talked one and we are able to do it again and again,just to make everybody see what we see.)

Thanks are due to Professor A.S.Troelstra for listening to and commenting on my (hasty) views on solipsism in mathematics (pp. 7 et sqq. below).In a more "philosophical" setting I would have been more careful in finding the due nuances when speaking about the "(in)dependence of mathematical results on a communication-process", taking into account the genuine (intuitionistic) counter-examples he has produced.The final form of the discussion was kept unchanged only because I still think it reflects accurately the "philosophy" behind the AUTOMATH Project.

I am grateful to the Mathematical Centre which accepted this work for publication as a "Mathematical Centre Tract".

Finally, I am indebted to Mrs. Josée Brands (Eindhoven University of Technology) and Mr. P.A.J. Hoeben (University of Utrecht) for clerical help in the final preparation of the text.

C O N T E N T S

0. Introduction.

This work is concerned with the abstract structure of some representative formal languages in the family of mathematical languages AUTOMATH (cf. DE BRUIJN 70-02, 73-30, 73-34, 80-72, JUTTING 79-46, VAN DAALEN 73-35, 80-73). The basic analysis was mainly motivated by the need for a theoretical approach to a couple of open problems concerning conservativity situations in AUTOMATH and closely related formal systems. Soon it turned out the approach which was taken in analysis is more significant for the understanding of the nature of an AUTOMATH-language than was initially thought of.

Though essentially self-contained, the expository parts of the present notes do - in general - presuppose some minimal background of mathematical logic (as, e.g., first-order logic, set theory and lambda-calculus) and of what is usually called "finite mathematics" (viz. rather elementary fragments of combinatorics). For instance, some familiarity with the basics of (both "type-free" and "typed") lambda-calculus should certainly enable the reader to follow or to reconstruct, on his own, remote details of the "language theory" of AUTOMATH on which we will not generally insist (and which are copiously documented in recent monographs as BARENDREGT 81, KLOP 80, REZUS 81 and - essentially - VAN DAALEN 80-73). Otherwise, the main exposition is primarily intended for logicians and this is largely reflected in our terminological habits (diverging in several respects from standard terminology in AUTOMATH but very close to generally accepted ways of speaking in mathematical logic). Of course, mathematicians, computer scientists and philosophers having some basic acquaintance with the methods of symbolic logic are thereby expected to find the present text easily readable.

The paper is intended to make accessible the technical details of construction of the main languages in the AUTOMATH-family and we have often deliberately omitted comments of a more general nature (say: almost everything pertaining to what might be called the "philosophy of AUTOMATH" should be recovered from several informal expositions due to N.G. de Bruijn which are already in print; see the reports cited above). Still, what is following below does not overlap (save for minor details) material discussed in VAN DAALEN 80-73, but information missing here can be, *mutatis mutandis*, safely retrieved from the latter work (which is highly recommended as a further advanced text and which is - anyway - the only place we are able to indicate so far where some proofs of important facts on the "language theory" of AUTOMATH can be found).

In this introduction we shall briefly review - still informally - the information needed for the understanding of the main text insisting on several aspects of the AUTOMATH project which have been insufficiently documented in the existing literature or have been simply discarded (due to mere lack of time and man-power or to deeper,"doctrinal" reasons).Also the relation of the present work to technical expositions concerning the standard "reference" AUTOMATH-languages will be made precise in the sequel.

01. AUTOMATH: historical landmarks.

The main ideas behind the AUTOMATH project (or "program"[1]) of formalization of mathematics go back to work done by N.G.de Bruijn in the mid-sixties (1966 or so) at the University of Technology in Eindhoven (The Netherlands).The first "public" report on the subject is seemingly the "preliminary study" DE BRUIJN 67-16,which is technically out-dated but still historically relevant,for some of the main objectives of the project were readily explicit there.

Before 1973 the only technical report describing in detail the "Classical AUTOMATH Language" (initially called "AUTOMATH";nowadays:"AUT-68") was DE BRUIJN 68-01. A sub-language of AUT-68 called PAL (= "Primitive AUTOMATH Language") - of main concern in this paper - was also isolated in this pioneering report together with some other "fragments" of AUT-68 which ultimately remained of a mere heuristical interest in standard presentations of AUTOMATH.

De Bruijn's PAL (which will be called sometimes here "PAL-THE") is not to be confused with its homonym "PAL-MIT" (say),which is a programming language - actually,a derivative of Peter Landin's ISWIM (= "If You See What I Mean" ;cf. LANDIN 66) - and was designed by a group of computer scientists at M.I.T., Cambridge (Mass.),in 1968,for teaching programming linguistics.So,in Cambridge (Mass.),"PAL" would mean quite different a thing from what it usually means in Eindhoven (TH-WSK),namely there it used to stand for "Pedagogic Algorithmic Language" (see EVANS 68,68a,70,72 and,possibly,WOZENCRAFT & EVANS 71).Here, the label "PAL-THE" means,simply,(de Bruijn's) "Primitive AUTOMATH Language" (cf. DE BRUIJN 68-01,73-30,80-73,etc.).

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1) It is perhaps more appropriate to characterise AUTOMATH as being a program of formalization of actual mathematics than to use the slightly anachronic label "project".Indeed,on the one hand,sufficient evidence has been produced in the meantime allowing to say AUTOMATH is a "grown-up" subject surpassing the stage of a mere "project" (in at least some acception of the word).On the other hand,

A (slight?) extension of AUT-68,called AUT-QE (= "AUTOMATH with Quasi-Expressions") was proposed in 1969 and - so far - this is the only AUTOMATH-language completely implemented on a computer,in Eindhoven (TH).

Due to some unpleasant features of AUT-QE,N.G. de Bruijn recommended in 1978 the study of a sub-system ("restriction") of AUT-QE called AUT-QE-NTI (= "AUT-QE Without/No Type Inclusion").The latter is not an extension of AUT-68 (though it extends PAL-THE,as expected),but there is some evidence to the effect that AUT-QE can be already "represented in" or "mimicked by the means of" AUT-QE-NTI.(See DE BRUIJN 78-56,for details and our discussion in 22. below.)

A complete formal description of AUT-QE (and therefore AUT-QE-NTI) was given by D.van Daalen in VAN DAALEN 73-72.Since then comprehensive pieces of actual mathematics have been translated into AUT-QE (and checked on computer).See JUTTING 76-60 and the underlying methodology described in JUTTING 79-46 or UDDING 80-69 for an alternative presentation of the theory of real numbers in AUT-QE.

The AUT(OMATH)-languages mentioned earlier (viz. PAL-THE,AUT-68,AUT-QE and AUT-QE-NTI) will be studied in detail,in an abstract setting,below.

Besides these there is a sub-family of AUT-languages (which we will generically call here LAMBDA-AUTOMATH) proposed by R.P.Nederpelt and N.G.de Bruijn in 1971 (cf. NEDERPELT 71-21,71-22,72-26,73-31 and DE BRUIJN 71-20,77-52b,VAN DAALEN 80-73, Chapter VII).These have been studied mainly for theoretical purposes and do not admit of "natural" interpretations (as it is the case with AUT-68 and AUT-QE). Still,AUT-68 and AUT-QE-NTI can be naturally embedded in any suitable version of LAMBDA-AUTOMATH and some version of the latter may be viewed as a proper extension of AUT-QE-NTI.
LAMBDA-AUTOMATH will be not studied in this paper.

(continued from previous page)

some of its objectives and prospects would perfectly justify a comparison with similar far-reaching entreprises (as,e.g.,"Hilbert's program","Curry's program" - cf. SELDIN 76/77,80,- "Church's program" - cf. REZUS 81 - or,why not,Bourbaki's). There are however many dissimilitudes in such a comparison: AUTOMATH has a rather weak foundational claim in comparison with any of the "programs" named above,while,inasfar feasibility is concerned AUTOMATH remains still one of the most convincing among these "programs".

A different extension of PAL, called AUT-Pi, was proposed by Jeffery Zucker (cf. ZUCKER 75-42 for an informal introduction). This seems much easier to write than AUT-68 (and even AUT-QE), but its "language theory" is considerably more complex than that of other AUT-languages. D. van Daalen has subsequently worked out the main language-theoretical details of AUT-Pi in VAN DAALEN 80-73, Chapter VIII (for problems which could not have been solved so far see op.cit., VIII.4.1.).

In the list above all AUT-languages but AUT-Pi are elementary - and so are some derivatives and modifications thereof which will be mentioned later - in the sense their only "constructors" are the head-constructors, the applicator and the abstractor (see 10. and 121. for details). AUT-Pi is not elementary in this acception and a detailed description of its "grammar" is a tedious affair diverging in some aspects from the "definition scheme" of PAL, AUT-68, AUT-QE, etc. For this - and some other reason - we shall examine AUT-Pi elsewhere.

AUT-Pi has been used (by J. Zucker and A. Kornaat) in order to write extensive pieces of "classical" analysis. (A long manuscript on Real Analysis has been produced which is not the "translation" of some text already existing before in natural language presentation. In fact, the language which was actually used in this manuscript is AUT-Pi-SYNT, an extension of AUT-Pi - cf. below -. It allowed a very "fast notation" close to the mathematical every-day language.).

Little has been done so far on the implementation of AUT-Pi (and AUT-Pi-SYNT) on a computer.

There are still two other distinct families of extensions of the AUT-languages named above. Both concern the formalization of some aspects of the metalanguage ("epitheory") of actual pieces of (mathematical) texts (or, better, of what would become "metalanguage" in standard formalizations of mathematics - not in AUTOMATH say).

In one direction (relying on suggestions going back to N.G. de Bruijn) one would want to extend any AUT-language by the so-called string-and-telescopes facilities (cf., e.g., JUTTING 79-46, 4.1.3. or ZUCKER 75-42 for an informal presentation of the subject). Such extensions, which we will call, for further reference, AUT-ST-languages, are very useful in the formalization of abstract structures (groups, rings, vector spaces, etc.) and have been actually used by J. Zucker (in the form of AUT-Pi-ST and AUT-Pi-ST-SYNT, cf. below).

Another family of extensions (initially motivated as auxiliary "input languages" for AUT-68, AUT-QE, etc.; cf. DE BRUIJN 72-25) arises by the addition of formalized "syntactic facilities" to (proper) extensions of PAL; hence the generic name "AUT-SYNT" for the languages in this (sub-)family. Any AUT-SYNT language incorporates,

qua formal ingredients,(i.e.,in the "language definition") syntactic pre-defined functions (these would appear as recursive epi-functions acting on the syntax of the language,in standard formalizations of mathematical texts and even in some appropriate formalization of an AUT-language).One of the most important features of these functions is that they can be calculated "mechanically".Incorporating them in the "language definition" of some AUT-language allows to omit tedious repetitions of otherwise redundant parameters (cf. JUTTING 79-46,Appendix 9.AUT-SYNT., for an informal explanation).
The "language theory" of AUT-68-SYNT and AUT-QE-SYNT was worked out in detail in JUTTING 82-83,while a somewhat more complex version of AUT-Pi-SYNT was actually used by J.Zucker and A.Kornaat in formalizing "classical" analysis (cf. below).

The AUT-ST and the AUT-SYNT extensions are practically independent from each other, but combinations of these facilities starting from the same "basic" AUT-language are still compatible (and practically very useful).Such combinations (AUT-ST-SYNT-languages,say) lead to considerably efficient systems of notation and writing actual pieces of mathematical texts in these extensions amounts to formalizations that are very close to the every-day mathematical practice (having also the advantage of allowing an automatic verification of correctness).
As noted earlier,J.Zucker was able to use AUP-Pi-ST-SYNT in writing directly an extended piece of text (on Real Analysis) without using the natural language as an intermediary step (in axiomatic formal presentation).

However,a shortcoming of the latter kind of extensions (especially the AUT-SYNT-languages) consists of the fact they lead to complications in both the underlying "language theory" and in the corresponding work of the actual verifier.L.S.van Benthem Jutting is currently working on the implementation of the languages in the AUT-ST- and the AUT-SYNT-family.(A verification-program for AUT-68-ST-SYNT and AUT-QE-ST-SYNT can be found in JUTTING 82-83.)

Besides the (sub-)families of AUTOMATH-languages named above,it is also worthwhile mentioning the AUT-4-languages (proposed by N.G. de Bruijn in 1974;cf. DE BRUIJN 74-44) and various extensions of AUT-68 and AUT-QE - the so-called AUT^{+}- and AUT^{*}-languages - proposed and studied in some detail in VAN DAALEN 80-73.

Roughly speaking,the languages in the (sub-)family AUT-4 are "segments" of appropriate versions of LAMBDA-AUTOMATH (where the maximal "degree" of correct expressions is restricted to 4 - hence the label "AUT-4" -;see VAN DAALEN 80-73,etc. for details concerning "degree-considerations" in AUTOMATH).
They were mainly motivated by an attempt to separate in a more straightforward manner matters concerning the "construction of objects" in AUTOMATH from those

concerning proofs of statements "on objects" (without,e.g.,manipulating intermediary "proof-classes").
However,only the "proof-part" of AUT-4 admits of a natural interpretation,whereas the corresponding "object(-construction) part" has,*mutatis mutandis*,the same disadvantages - as to a natural "semantics";cf. also 03. below - as any language in the LAMBDA-AUTOMATH-family (viz. the correctness rules of AUT-4 would also allow "ultimate objects" - as the naturals say - to be "inhabited",almost in the same sense "types" are allowed or stipulated to be "inhabited").
There is some hope that this unpleasant feature of the AUT-4-languages could be circumvented by establishing appropriate "conservativity"-results (see DE BRUIJN 74-44), but nothing has been done so far in this direction and no AUT-4-language has been implemented until now.

On the other hand,van Daalen's extensions AUT^{+} and AUT^{*} were introduced for theoretical purposes,while studying the epitheory of AUT-68 and AUT-QE (mainly the corresponding "closure properties",stating the invariance of correctness under reduction).In general,such extensions are conservative over the corresponding "non-+" or "non-*"-languages,but the "structure of correctness" is more "regular" inasfar the former are concerned.
As incidentally noticed by L.S.van Benthem Jutting,it might turn out some of van Daalen's extensions would also admit of easier and more economical implementations (as regards the work done by the corresponding verification programs).

Despite their diversification,it is certainly possible to establish general metatheoretic standards for the description of the languages in the AUT-family.
Some effort has been done in this direction since 1974 (cf.,e.g.,DE BRUIJN 74-40, 77-52a,b and,of course,VAN DAALEN 80-73),but a satisfactory,unifying metatheory - embracing all relevant aspects of the empirical work on AUTOMATH done so far - is still lacking.

The metalanguage and the description-scheme proposed in this paper were actually intended to cover only what we would call the elementary AUT-languages (cf. above; in this sense,both AUT-4 and van Daalen's extensions are elementary,while AUT-Pi, the AUT-ST,the AUT-SYNT-languages as well as the possible combinations thereof are not so).
It shouldn't be too difficult a task - however - to adapt the present standards such as to function as a general description-frame for all members of the AUT-family.

02.AUTOMATH: tracing back motivations.

Mathematics is (pace Brouwer and some solipsists) a social - "inter-subjective" say - affair.What a mathematician claims he "sees" or thinks is true does always deserve some proof,no matter what are the professional abilities or the reputation (qua mathematician) of the person making such claims.
Fortunately,any proof of a mathematical theorem is objective at least in the following sense: if some mathematician claims he has a proof of some theorem (on number theory say) in his mind then there should be some other mind (a mathematician's mind) able to "grasp" or to understand and,eventually,to reproduce the "structure" of the claimed proof.This "epistemic event" (proof-discovery,on the one hand and proof-understanding on the other) should sooner or later result in some objectively perceivable fact,exemplifying "communication of mathematical ideas".
There is no "mathematics" beyond this least,paradigmatic,"inter-subjective" (social) intercourse and some form of language must always underly the latter.
As puzzling as it might appear,mathematics does not exist before and in complete independence of some communication-process.Its practice as a "solitaire"-game is always an elliptic condensed form of language-use.

The only serious point in the solipsist's objections and arguments concerning the communication of mathematical ideas/results (cf.,BROUWER 81,pp.25-35,on Mathematics and experience ,the fragments of Brouwer's dissertation published in VAN STIGT 79,BROUWER 07 and,possibly,more recently,ZANSTRA 71) is that an absolutely certain communication of ("actual") mathematical ideas ("proofs",etc.) is a very difficult practical problem.In fact,"absolutely certain" should be always cautioned as being too strong a requirement in any practically given communication-situation.In practice,not everything a mathematicisn may have in his mind (a whole system of intuitive interpretations and references to "models" found previously) is relevant for the content of some proof he claims to "have" and intends to communicate.

There are different ways in which one might become aware of the difficulties mentioned above.

(A) First,very often mathematicians are used to and like to practice an elliptical way of speaking.
The mathematical ellipsis can be of many kinds (at least in colloquial speaking). However,written texts offer some uniformity: it won't be too difficult to get a possible "standardization" of the ways it is used.For present purposes we shall

essentially distinguish between two such kinds of uses (other uses could be possibly discovered by "empirical" investigation but we suspect they will be either mere variants of those mentioned below or - still - will be rather irrelevant in the present context).

So between a label *Proof.* and some other label (possibly abbreviating "Quod erat demonstrandum."),indicating the end of the announced proof,a professional mathematician would often want to write either (1) "Obvious.","Evident.","Straightforward." or some lexical variant of these or (2) "Standard.","As earlier.",etc.[1)]

Now a "proper" proof is always "a kind of composite reference to previous proofs" (N.G.de Bruijn) and "filling in" a gap in some proof (or,which is the same,eliminating some elliptical way of speaking in favour of an explicit,"complete" one) would amount to some recursive search throughout existing mathematical texts (or, more properly,throughout an existing mathematical practice),which may be,again, "elliptic" in nature.The process should always stop somewhere in some "principles" or "axioms" ,viz. at statements one would rather want to accept without proof . So the activity of understanding a given mathematical text (i.e.,proof-understanding which is,in the end,a kind of proof-(re-)discovery) can be safely described as possibly being a kind of tree-like search,where the initial clauses of the underlying recursion are a matter of common agreement ("truth by convention" say,to use differently a popular phrase in analytic philosophy).

To restore the proper "composite reference system" implicit in some mathematical ellipsis of kind (2) above is - in most of the cases - comparable with the "trivial" work done by a bibliographer or "documentalist".Of course,this ultimately depends on one's mathematical experience or knowledge ;still one would hardly need some "invention" ("proof-discovery") here.The "hard work" was readily done once (some time,somewhere,possibly by some other people): if some proof is "standard" all we have to do in order to recover its due "reference system" is to supply some "footnoting" say, containing the actual references to the existing literature (the so-called "mathematical folklore" may be also therein included).

So,provided it is not exclusively concerned with the "foundations" of some particular topic - written up in its very details - a mathematical book is always longer than as it appears (in its "concrete",physical format).Or,properly speaking,it should be so,when "gapless" proofs are intended.

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1) In the former case some would also write "Omitted." (which has sometimes the second sense),while in the latter case one may encounter even "As ever."

Hovewer,it is not so easy to supply details of proof in case of elliptical indications of the form "Obvious.",etc.(under (1) above).Indeed,"obvious" and its lexical variants - when taken as proof-qualifiers[1] - are,to use a word of B. Russell,"indexical expressions" (like "I","today","here" or even "this country"[2] say;cf. MONTAGUE 68,70,etc.).

That is to say: the actual "meaning" of a proof-qualifier of the kind "obvious" (1) may depend on many pragmatically unobvious parameters,as,e.g.,location in time, personal mathematical experience (cf. with "I think that...","I believe that..." expressing propositional attitudes),etc.
In any case,one should take "obvious" - qua proof-qualifier - as qualifying some proof that could have been displayed (in full) and only by some abuse of (meta-)language as qualifying the corresponding statement to be proved (phrases like "the statement ... is obvious" are,therefore,to be interpreted as derived uses of "obvious" as a proof-qualifier).

In other words,"obvious" (and its lexical variants) would rather qualify some (finite) sequence of steps ("actions") to be performed in order to recover a system of - possibly nested - references that can be - at least in principle - described recursively.(The latter description should be - of course - a piece of metamathematics or - to use a word of H.B.Curry - of "epi-theory".)

The underlying necessary tree-like search won't be - in this case - exactly comparable with the documentalist's search in a Universal Mathematical Library: the lat-

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1) "Obvious" may also occur,incidentally,in colloquial ways of speaking more or less related to mathematics when it is used to qualify some state of one's private (i.e. geometrical) intuitions.Then it is not used as a proof-qualifier. Indeed,if some under-graduate student (in mathematics) claims a given rotation in 3-d space leaves the set of edges of a given cube invariant and he makes the additional claim that this very fact is obvious for him (without being able to display a proof for the first claim) then his use of "obvious" is certainly different from what is meant here.We should therefore carefully distinguish between "i-obvious" (= "intuitively obvious") and "p-obvious" (= "obvious qua proof"),while even if both expressions are indexical in the mathematical everyday language (MEL or WOT,in Dutch),we should consider the first use as being metamathematically (here "proof-theoretically") irrelevant.(The distinguo mentioned here and the corresponding example are essentially due to N.G. de Bruijn, in conversation.)

2) In cases where the speaker/writer is not located in the United States (it seems U.S.A. is the only country where "this country" is not an indexical expression).

ter kind of work should certainly presuppose a minimal amount of "invention" ("proof-discovery").
Still,if "obvious" is properly used as a proof-qualifier,the needed "invention" should concern rather elementary steps or sequences of steps - "sub-trees" in a tree-like search - that can be eventually (re-)discovered by any other mathematician (provided he is sufficiently informed in the particular subject matter of concern).

In the end,the first kind (1) of mathematical ellipsis noted above is not essentially different from the second one (2),inasfar the form of eliminating each of them is concerned.Both kinds involve ("refer to") some common background of mathematical experience ("common" to a given group etc.) and differ only in the ways this experience is invoked or referred to.Anyway,in both cases some pragmatic context has to be recovered in order to understand what should/might stand for a given mathematical ellipsis (in some particular - mathematical - text).

(B) Another way in which one may become aware of the difficulties involved in the process of communication of mathematical ideas is to acknowledge the possible disagreements as to the use of what should be an "acceptable/good/correct argument".

In this case,what is questioned is not the completability of a given elliptic argument (on which topic agreement may still subsist),but rather the nature of the steps involved in such completions or,even,the "logical form" of readily complete(d) arguments.

This is,again,a metamathematical subject.We have at hand proofs "as a kind of composite references to other proofs" and we are free to handle these references by a system of operations which is,essentially,the same as that used to handle "objects",mathematically.Moreover,we can do this without thereby being commited to some form of Platonistic ontology or without seriously thinking of these proofs as being actually objects (N.G.de Bruijn).In this way,we should be able to identify ("single out","isolate") the exact portions of the proof-/reference-system on which disagreements may occur and to accept or reject the possible criticism concerning the use of these "segments" as proper parts of given proofs.
In particular,the corresponding system of operations used to handle proofs (qua metamathematical "objects") has to be chosen such as to allow an unambiguous understanding of the rôles played by each primitive notion,axiom,definition or (sub-)proof in any argument/proof submitted to such a criticism.

(C) Even if we have at hand a rather complete text of mathematics and we do not intend to submit to any criticism whatsoever the patterns of reasoning used therein (i.e.,we do not question its underlying "logic") some difficulties may still appear on the way we intend to use this text;these might concern say the efficiency of our way of reaching some particular result appearing at some place in it.

Indeed,suppose we are interested in some given theorem occurring at some "advanced stage" in a mathematical text.If this happens to be conceived (by its author) as a comprehensive treatise on some subject and we only want to recover the information necessary for the understanding of that very theorem then reading the text from the very beginning (faithfully following the author's organization of the material in the book) is not always the best strategy to adopt: indeed,the author may have inserted a couple of Chapters in the book,preceeding the statement of the theorem we are interested in,and the proof of the theorem may not "logically depend" on material presented in these Chapters.To use the text efficiently (which is,again,an aspect of the process of communication of mathematical ideas) means,in this case,to have at hand some reasonably manageable system of processing it,possibly allowing us to omit those portions of text which are "redundant" according to our momentary,"local" interests.

In any of the situations mentioned above,it is clear that an "absolutely certain" communication of mathematical ideas - viewed as a practical problem - presupposes,first of all,the existence of a good formal language.

Such a language must be viewed - "globally" - as a means to describe the actual practice of writing mathematical texts and thereby it should meet several basic requirements,viz.

(a_0) to allow writing up actual proofs in a very detailed manner such as to avoid any possible elliptic formulations yet bearing a close relationship to the ordinary mathematical practice;

(a_1) in particular,the formal presentation of a mathematical text in such a language should straightforwardly allow some form of inter-subjective (or,if one prefers:"objective") verification of the correctness of the original text (e.g.,to unify the uses of "being obvious",*qua* proof-qualifier,up to a standard one),

(a_2) by possibly carrying out "correctness-checking" in a pure automatic way (*via* some implementation on a computer say);the idea of a computer-assisted proof-checking of actual mathematical texts is central in the AUTOMATH-program of formalization of mathematics (cf. DE BRUIJN 67-16,68-01,69-17, 69-18,70-02,73-30,73-34,76-43,80-72;NEDERPELT 70-19;VAN DAALEN 73-35,80-73, JUTTING 79-46;ZANDLEVEN 73-36;ZUCKER 75-42);

(b_0) to be logically (and,in general,philosophically) neutral;i.e.,not to commit ourselves to some particular set of assumptions as regards the "acceptability" of the patterns of reasoning actually used in the text to be formalized or to some set of ontological presuppositions,possibly conflicting with rival formalizations (differently motivated from a philosophical standpoint);

(b_1) in particular,such a formalization should straightforwardly allow the understanding and (the) analysis of the structure of any particular ("gapless") mathematical proof,no matter which is the "logic" adopted/accepted as "basic" in proof,

(b_2) being such as to single out the rôles of primitive notions,axioms,definitions,"rules of derivations",proofs and theorems such as to make transparent any possible metamathematical criticism as to the use of each such an item in particular mathematical texts,

(b_3) and to assist us in understanding the complexity of particular (mathematical) arguments (i.e.,to reveal the possible analogies in the structure of arguments and help us to classify these patterns of proof according to their intrinsic difficulty).

(For details see,e.g.,DE BRUIJN 73-30,73-34 or 80-72,4.Understanding.)

(c_0) Finally,the structure of the intended formalization(s) should be such as to allow us a convenient way of storing and processing the information present in a large number of mathematical texts;

(c_1) the ideal situation to reach would consist of having at hand a kind of comprehensive encyclopaedia ("data bank") of mathematical results capable of

- being stored conveniently (together with the corresponding proofs),in a suitable "information system" (a "Universal Mathematical Library" say)

and of

- being processed (selected,retrieved) according to local or momentary needs,as,e.g.,
  - displaying a glossary of particular texts ("lists of definitions")
  - locating the rôle - if any - of a particular primitive notion or axiom in some given argument/proof,
  - "excerpting" a given text containing some particular statement such as to avoid any "redundant" details necessary for the understanding of its proof (or for its understanding,tout court),

- identifying the "minimal logic" necessary for the proof of some given theorem,

etc.,etc.

(For details see,again,DE BRUIJN 73-34,80-72,etc.)

The requirements listed above are - in principle - satisfied by most of the existing languages in the AUTOMATH-family.
Actually,almost all these requirements were among the motivations which have led N.G.de Bruijn to propose and work out in detail the "Classical AUTOMATH Language" AUT-68 (see DE BRUIJN 68-01).
Subsequent work done within the AUTOMATH research group since 1967 was mainly intended to <u>refine</u> the initial ideas up to a set of <u>optimal solutions</u> meeting the specified objectives.

Paraphrasing N.G.de Bruijn (cf. the <u>Preface</u> to JUTTING 79-46) one might say that the existing work on the AUTOMATH-project has - by now - at least some undeniable <u>historical significance</u>:

"...never before has a substantial piece of mathematics been presented on a comparable level of completeness and precision",within some formal language.

(Usual comparisons with Peano's <u>Formulaire</u> or with <u>Principia Mathematica</u> do always miss one or another aspect: Peano lacked details of proof while <u>Principia</u> reached only very elementary portions of actual mathematics in formalization.)

03. AUTOMATH versus semantics.

One of the motivations underlying the work on AUTOMATH was (DE BRUIJN 73-34) "...to make something of a universal nature".

Even from a superficial review of results obtained since 1968 in this area, it should be clear that the claimed universality is to be understood in some immediate (non-philosophical say) sense : the project is feasible at least in the sense that large parts of actual ("ordinary": i.e., "classical" as well as "constructivistic") mathematics can be presented in AUTOMATH in a natural way. (This is a "weak foundational claim" as we may learn from VAN DAALEN 80-73, I.1.6. etc., and, as the way of understanding "formalization" is quite different from that intended in mathematical logic or set theory, there is no possible "conflict" with Gödel's results on incompleteness.)

It is not our purpose here to estimate how "large" could possibly be the parts of actual mathematics that can be fed into and checked by means of an AUTOMATH-verifier (the extant work done in Eindhoven - TH - might offer ad hoc, experimental estimations of the kind: "at least ZF" or "at least Classical Real Analysis", etc.). Rather we may notice that this claim of universality might be of some immediate interest for logic and metamathematics; viz. it is a fact that, when provided with suitable semantics, an AUTOMATH-language would become a useful tool in building up a general theory of constructions and proofs for mathematics (whether "constructivistic" or not).

Unfortunately, the model theory of (the languages) AUTOMATH is a rather undeveloped topic.

The lack of interest for semantical and model-theoretic investigations in AUTOMATH is certainly rooted in the philosophy behind the project.
One of the most common attitudes in this respect (shared by several contributors to the subject) is summarized as follows: a piece of actual mathematics is already a "model" for some correct AUT-text and this "intended model" is somewhat self-sufficient for its "understanding".
So there is no point in considering something like "the class of all models" for a given correct AUTOMATH-book and we should be even less interested in the (huge) class of all models for the set of all correct AUTOMATH-books (the latter would be "isomorphic" to the class of all pieces of actual mathematics that could be ever written, if we have to take seriously the claim of universality referred to above).

Actually,"mathematical semantics" or model theory have to be understood as pieces of actual mathematics and - as such - they would again require "correctness proofs" (possibly in the AUTOMATH-system).This would unnecessarily complicate the matters without actually improving our way of understanding AUTOMATH.

Another aspect of the common claim that AUTOMATH needs no semantics is to be located in the constructivistic component of what we may call "the philosophy of AUTOMATH" (i.e.,the ideas and conceptions on the nature of mathematics and its language underlying the project AUTOMATH): indeed,the methods currently used by model theorists are - in most of the cases - infinitistic in nature.Such methods won't fit any more the metamathematical standards accepted as admissible in the description and the study of the AUT-languages (which are - at least in intention - "machine-oriented" in the sense they have to be "accepted" by some computer).

The situation is,roughly,comparable with the (controversial) status of the "classically fashioned" semantical investigations into Heyting's logic (see,e.g., TROELSTRA 73,GABBAY 80 or VAN DALEN 8[illegible] for surveys): the latter are not "intuitionistically acceptable" from the standpoint of the Brouwerian tradition.

Now,the attidude described above is certainly open to some criticism;to keep in force,for a while,the parallel with intuitionism,suggested above,one would want to note that a "classically-minded" mathematician would safely understand abstract "classical" structures as what we are nowadays used to call "Kripke models" (or "Beth-Kripke models") for theories based on Heyting's logic and even take some profit - metatheoretically - from this understanding.
Similarly,there is - in principle - no prohibition against the fact that some class of mathematical structures may provide an abstract description (relevant for understanding say) of the every-day practice of writing "correct mathematical texts".
This class (if any) may turn out to be completely uninteresting from the point of view of someone sharing basic philosophical convictions kindred to the philosophy of AUTOMATH;still,there is no a priori reason to forbid the search for such (a class of) structures,if - above all - successful work in this direction might turn out to be useful for a metatheoretical investigation of the AUT-languages.

In absence of a sound model-theoretic basis for the AUTOMATH-languages it is not surprising that what we actually know about their metatheoretical properties is rather scarce.

Indeed,the existing work on (the languages in the) AUTOMATH (-family) was,until now,syntactically oriented.The main contributions to the topic (labelled within the project as pertaining to the "language theory") consist,up to date,only of a few facts concerning the behaviour of "correct expressions" under reduction (the so called Weak/Strong Normalization theorems,Church-Rosser theorems,"Closure"-theorems,stating the invariance of the correctness categories of a given language under the underlying notion of reduction;cf.,e.g.,JUTTING 71-23,71-24;NEDERPELT 73-31;VAN DAALEN 80-73).

One has,however, - so far - no means to estimate the "proof-theoretic strength" of the AUT-languages and, in general,no means to make comparisons with any other (differently oriented) formalization.
There is,of course,the straightforward approach which would consist of "translating" - tale quale - into some AUT-language rival formalizations as to their "expressive power",but this is rather global and uninformative a way of doing things. ZF can be certainly "phrased into" AUT-QE - the work was done by D.van Daalen,as an exercise;cf. VAN DAALEN 70-14,70-15 -,but this does not mean that AUT-QE and ZF are "equally strong" as regards their proof-theoretic properties. Say: are there results that are "unintended" in ZF which can be "derived" within the corresponding AUT-QE-book ("formalizing" ZF)? Similar questions may be raised in connections with rather different formalizations (whether "foundationally"-oriented or not).

Among other things,the lack of (abstract) semantics has largely contributed to the "parochialization" of the AUTOMATH-project.
With another historical parallel: the type-free lambda calculus and the related theories of "combinatory logic" remained mere technical curiosities in mathematical logic - after Church has completely abandoned his foundational program (cf. CHURCH 1936;for more details see REZUS 81),despite the work done in the school of Curry (CURRY et al. 58,72;HINDLEY et al. 72,etc.) - until D.Scott discovered (in 1969) an appropriate class of mathematical models for these systems (SCOTT 69; see BARENDREGT 81 for a survey and REZUS 82 for further references).
Almost the same thing happened with the Relevance Logics (ANDERSON & BELNAP 75, ROUTLEY & MEYER 8X,etc.),which got a large audience only due to model-theoretic work (of J.M.Dunn,R.Routley,R.K.Meyer,K.Fine,A.Urquhart,etc.)

Besides the philosophical reasons noted above,the situation complained here has also a deeper motivation in the difficulty of the subject: the languages in the AUTOMATH-family were (and still are) resisting to a model-theoretic approach because of their intrinsic complexity.The underlying type-structure is - with a word of D.van Daalen (VAN DAALEN 80-73) - a "generalized" one: one has a "dependent type-structure" where the "typing expressions" and the "typed expressions"

are to be generated simultaneously,as in the case of the languages proposed by Per Martin-Löf for the formalization of constructive mathematics (cf. MARTIN-LÖF 71,72,75,75a,75b,79;ACZEL 77;BEESON 80;for a differently oriented use of closely related languages see CONSTABLE 80).
This difficulty was also incidentally noted in BALSTERS 82.

On the other hand,model-theoretic work on the (simply) typed lambda calculus and the related theories of functionality has been available only recently and is, as yet,unpublished (BEN-YELLES 79;BARENDREGT *et al*.80;HINDLEY 80;etc. confirming conjectures of D.Scott;the main proposals for models are also due to D.Scott).

The present author shares the opinion that some model-theoretic work on the languages in the AUTOMATH-family is *necessary* in the present status of research. *Syntactic methods* - if somewhat involved - have already played their rôle in the game and shown their strength.It is not too much to be expected in this direction. A recent proposal for *a mathematical semantics* of the main AUT-languages will be discussed in detail elsewhere (cf.BARENDREGT & REZUS 80).

Even if this paper still *essentially relies on a syntactic analysis of correctness*, the approach taken here is somewhat differently oriented in comparison with work done so far on the subject.
Specifically,we are not interested in "local" features of one or another language in the AUT-family (as,e.g.,their behaviour under reduction,w.r.t. conversion = "definitional equality" or so,which would depend on the *exact formulation* of the correctness rules of a particular AUT-language),but rather aim at an understanding of the abstract mathematical structure of any AUT-language *qua* syntactic object.(The outcome might also shed some light on the possible place to look for mathematically interesting models for AUT-languages.)

It is certainly pleasant to discover that no AUTOMATH-language is sensitive to some particular formulation of it.Still,the actual "reference"- or "standard"-formulations are somewhat too "practically oriented" (or "machine-dependent") in order to be accepted as a proper object of metamathematical study.

This,finally,motivated our *abstract approach* below.

04. The contents of this work.

We distinguish, *prima facie*, between the "reference"-version of an AUT(OMATH)-language – without insisting too much on the different levels of use of the "reference"-concepts[1] – and the corresponding "abstract"-version.

The latter versions convey – essentially – the same kind of information as the "reference"-versions, but can be described accurately in a more natural way and are subject to a more convenient metamathematical manipulation. (It is easier to "speak about" them, and this shouldn't be too surprising, for the "reference"-versions were devised such as to be easily implementable and not in view of a pure formal study.)

There are easy-to-find transformations from the abstract-AUTOMATH languages – as presented here – to the "reference"-counterparts and backwards, which should be obvious to the reader who has some previous experience with the latter.
Anyway, at a later stage in exposition, it will possible to "define" – within the metalanguage used to describe the "abstract"-versions – almost all concepts that are used in the "reference"-description (without actually duplicating the work).

Roughly speaking an abstract AUTOMATH language must be thought of as being the "core language" from which its "reference"-version may arise by "sugaring"[2] the "theoretical syntax".

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1) To mention only the most important distinctions: any AUT-language has a "standard reference version", as specified in the language definition and a "physical version", as used in displaying actual pieces of AUT-text, whereas the latter level may be viewed either as a "publication language" – as used by the translator – or as a "machine-oriented" language, – as processed by a computer instructed to check the correctness of a piece of "physical" AUT-text.
On the other hand, the "standard reference version" can be specified by different language definitions. E.g., there is an "E-definition" (VAN DAALEN 73-35, 80-73), which will be used as a starting point in the present description, and an "algorithmic definition", which is very close to a program checking correctness of AUT-books. Other, more pedagogic, presentations – using a "natural deduction"-style (*à la* F.B. Fitch, say; cf. FITCH 52, MONTAGUE & KALISH 64, etc.) – have been preferred by N.G. de Bruijn (DE BRUIJN 68-01, 73-30, etc.).

2) With a word of Peter Landin, the "syntactic sugar" contains features that are added to a given formal language in order to facilitate its use without increa-

So,properly speaking,our considerations do not apply directly and ad litteram to things like PAL(-THE),AUT-68,AUT-QE,etc.,but to their abstract counterparts. For the latter some more economical nomenclature has been established (just to save symbols),still in close relationship with that in use for "reference"-AUTOMATH. The due minimal "dictionary" is as follows:

Abstract AUTOMATH	"Reference"-version
PA Primitive AUTOMATH	PAL(-THE)[1]
CA Classical AUTOMATH	AUTOMATH[2],AUT-68
QA	AUT-QE
$Q^{-}A$	AUT-QE-NTI
ΛA LAMBDA AUTOMATH	Λ[3]
ZA Zucker AUTOMATH	AUT-Pi
etc.	

The main peculiarities of the abstract AUT-languages over their "reference"-counterpart are to be located in the choice of the primitive syntactic categories.

For a "global" comparison we display here the correspondences abstract vs "reference"-AUTOMATH as regards this choice ("reference"-categories as in VAN DAALEN 73-35):

Abstract AUTOMATH:	"Reference"-version:
Terms	Expressions
E-sentences (E-formulas)	E-formulas
————————	Q-formulas
Variable-strings	————————
Term-strings	————————
Contexts	Contexts
Constructions	Lines
- ————————	- EB-lines (declaring variables)
- p-constructions	- PN-lines (constructing "primitives")
- d-constructions	- definitional lines (constructing "defined notions")
Sites (= finite sets of constructions).	Books (= finite sequences of lines).

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continued from previous page

sing its (semantic say) strength.

1) See 01. above.

2) Historical label (used in DE BRUIJN 67-16,68-01,etc.)

3) In "book-and-line"-format (cf. also VAN DAALEN 80-73).

For a detailed explanation of the description-scheme of the abstract AUT-syntax we refer to 05. below.

Here we give only some hints motivating the main differences.

(1) It is not necessary to have a primitive syntactic category Q-formulas in the language (anyway, such "formulas" will never appear as syntactic units in some "reference"-AUT book). Accordingly, they will be constructed - in the abstract setting - by some appropriate détour via the metalanguage.

(2) EB-lines are not essential as separate syntactic units (they can be appropriately "stored into" contexts; this approach was actually taken by D.van Daalen in VAN DAALEN 80-73). On the other hand, the other kinds of lines of the "reference"-version describe "constructions of primitive / defined notions" while the EB-lines are, rigorously speaking, "variable-declarations". Putting them together under the same syntactic rubric would somewhat break the uniformity of the syntactic description.

(3) The linear order in "reference"-AUT books is completely unessential for the description/understanding/study of correctness in AUTOMATH. What actually matters in this respect is a different kind of order (a partial order) describing the "reference-system" of a given PN- or definitional line (i.e., what may be called, "reference structure" in an AUT-book).
Accordingly, we shall keep only a syntactic category of Constructions, containing
(i) p-constructions corresponding to the former PN-lines and
(ii) d-constructions corresponding to the definitional lines of the "reference"-formulation,
while, instead of considering finite sequences of such items, we decide to pay attention first only to finite sets of constructions (= sites).
The "real" reference structure in sites will be hereafter obtained by an appropriate analysis of the structure of correctness (cf. 3. below).

As a matter of methodology, the description of the syntax of abstract AUTOMATH will be conveniently separated into

(a) a "correctness - free" stage
(b) a "correctness" stage.

This can be done, for the AUT-languages of concern here, under some additional metatheoretical stipulations concerning the behaviour of the constants used in particular formalizations (what we called below "floating constants"; cf. 10. etc.). The approach would enable us to speak about "delta-reduction", etc. even at a "correctness-free" stage (whereas "delta-reduction", "definitional equality", etc. have, properly speaking,no meaning "outside" an AUT-book). Cf. 05. below.

The abstract AUTOMATH-languages studied in this paper are PA,CA,QA and $\overline{Q}A$. Section 1. is devoted to the description of the "correctness-free"-part of their syntax,while section 2. is concerned with a detailed presentation of the "correctness"-part.

The "global properties of correctness" make up the main content of section 3.

The latter concern as well as the preparatory - abstract - description were motivated by the study of a conservativity problem in AUTOMATH.

The motivating problem concerned the relation of AUT-68 and AUT-QE to the "least" language in the AUTOMATH-family: PAL(-THE) and it is - in itself - of some philosophical interest.Besides this,it has also a more immediate,practical import.

The philosophical aspect is more or less related to the epistemology of mathematics (as conceived say by Jean Piaget and his school of genetic epistemology in Geneva).Specifically,it concerns the development of the language of mathematics along its history.
To be more explicit,N.G.de Bruijn remarked that the conceptual apparatus of the (pre-)XVIII-th century mathematics (or,at least,the most part of it) can be expressed as well in PAL(-THE).
The idea of a function in general and the rise of mathematical logic (which are achievements of the XIX-th century) require more powerful means of expression: functional abstraction and the behaviour of quantifiers cannot be expressed in PAL(-THE),while they would admit of a proper formalization in AUT-68 (or AUT-QE). Establishing a conservation property - in the due sense - of AUT-68 (or AUT-QE) over PAL would lead to a straightforward estimation of the "degree of complexity" of a given mathematical text: e.g.,results that can be "phrased into" a relatively simple language (that of pre-cantorian mathematics say;to fix some arbitrary - not too remote - a landmark) can be already obtained with the ("historical") means of the mathematics promoting and actually using that language (here: with the means of pre-cantorian mathematics).
This claim is not exactly equivalent with Hilbert's assumption that any mathematical problem is solvable.Rather it establishes a close connection between the complexity of a given mathematical language (in its historical appearance) and the "depth" of the methods and concepts "presupposed" or "promoted" by this very language.[1)]

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1) One of the topics on which N.G.de Bruijn lectured (since 1978 on) at the University of Technology in Eindhoven concerns "The Language and Structure of Mathematics" (cf. DE BRUIJN 78-61 and the forthcoming book DE BRUIJN & NEDERPELT 8X).

The practical aspect of the conservativity problem mentioned above is somewhat easier to grasp and concerns the possible economy (in time and computer-memory) over the work done in the actual verification of an AUT-text on computer: it turns out a PAL-text is always somewhat easier to check than an AUT-68- or an AUT-QE-text.
Now if AUT-68 say is actually conservative over PAL(-THE) - in the due sense - one can choose to formalize in the PAL-segment as much as possible - from the very beginning - without thereby loosing "in strength" (and,by conservativity, any AUT-68-"line" that "can be written in" PAL - in the expected sense - and admits of a "correctness proof" in AUT-68 should also admit of a "correctness proof" in PAL).

The <u>concept of conservativity</u> - as relativized to the case of AUTOMATH-languages - which is actually used here is essentially that of DE BRUIJN 74-44.(See <u>30</u>. for heuristics and a precise analysis.)

At a closer examination,it turned out that somewhat <u>stronger</u> a property - implying conservativity,in the sense of de Bruijn - can be proved about AUT-68 and/or AUT-QE <u>versus</u> PAL-THE.By analogy with situations occurring in (first-order) logic we called it "PAL-separation property".Roughly speaking,it can be stated as follows:

PAL-Separation Property:

Let AUT be any one of the following languages: AUT-68,AUT-QE,AUT-QE-NTI,etc.
For any "correct" AUT-book $\underset{\wedge}{B}$ such that $\underset{\wedge}{B}$ contains some line k,if

(1) k is "correct" with respect to $\underset{\wedge}{B}$ and

(2) k is "written in" PAL

then there is a "correct" PAL-book $\underset{\wedge}{B}^{V}$ such that

(3) $\underset{\wedge}{B}^{V}$ is a "sub-book" of $\underset{\wedge}{B}$ (<u>qua</u> sub-sequence - of lines),

(4) k is "correct" with respect to $\underset{\wedge}{B}^{V}$

and,moreover,

(5) $\underset{\wedge}{B}^{V}$ is <u>the least</u> "correct sub-book" of $\underset{\wedge}{B}$,satisfying (3) and (4).

(Here $\underset{\wedge}{B}$ is "correct according to the rules of AUT",$\underset{\wedge}{B}^{V}$ is "correct according to the rules of PAL" and "written in PAL" is to be understood "modulo definitional expansion",viz. any expression which is a "component" of k has a PAL-"reduction graph". These notions will be made precise,in an abstract setting,in sections <u>2</u>. and <u>3</u>. below.)

The main work in 3. relies on a combinatorial analysis of correctness in sites. The abstract setting allows to isolate in a straightforward manner the relevant structure of correctness in sites ("correct" sites are actually posets with respect to the "definitional history" of the constructions they contain):this is the so-called "reference-order" (cf. 321.below).Relying on this order,a precise "measure of complexity" for constructions is introduced (cf. 322.).As the latter is given by some appropriate recursive function it can be used as a useful meta-theoretic tool in proving facts on corretness in abstract AUT-languages.Finally,the underlying analysis is - in a sense - "global",for it does not actually depend on the exact formulation of the "correctness rules" in one or another (abstract) AUT-language.So,in the end,the main results can be easily transferred to other AUT-languages,not actually taken into account and described in detail in this paper (as, e.g.,ZA - the abstract version of Zucker's AUT-Pi - versus PA,etc.)

Needless to say that the facts established here for the abstract AUT-languages hold as well for the corresponding "reference"-versions.

05. Abstract AUTOMATH: definition scheme.

The syntax of any abstract AUTOMATH language LA can be given by specifying:

(i) the alphabet A_{LA} of LA,

(ii) the syntactic categories of LA and

(iii) the correctness categories of LA.

(i) The alphabets A_{LA} are denumerably infinite sets of pairwise distinct structured symbols (i.e., sets of symbols for which some implicit categorization into - pairwise distinct - sub-alphabets is given). Cf. 10 below.

(ii) For each LA, the syntactic categories of LA are

- syntagmatic categories (from the Greek radical "tag";"syntagomai" = I am putting together, hence "taxis" = ordo and, of course, "syntaxis")
- text-categories, containing the finite sets of some "designated" syntagmatic category of LA; there is only one such a category in our presentation below, viz. the category of sites ($Site^{LA}$, for any LA, and $Site^{LA}$ is canonical in a sense to be explained later).

The syntagmatic categories of LA are generated in two distinct stages:

- the free stage, where the free categories of LA are produced; these will be given by recursive stipulations which concern only the well-formedness of expressions in LA, and
- the canonical stage, producing the canonical (syntagmatic) categories of LA; these are either identical to some of the preceeding ones or are obtained from these by elementary processes, as, e.g., taking infinite unions or finite sequences (pairs or triples say) of elements belonging to the free categories.

  There is only a finite family of canonical categories in LA and its elements are canonical in the sense that they are to be taken into account in the definition of the correctness categories of LA.

  The distinguo between free and canonical (syntagmatic) categories in some LA is also necessary in order to insure the finitary character of the process of generation of the correctness categories of LA.

05.1.REMARK.

There is also another difference between the free and the canonical stage of generation mentioned above;namely: the free syntactic categories of LA can be generated by a Montague-like grammar,under some unessential additions (see MONTAGUE 70a) and,up to a certain point,they would also admit of a context-free grammar ("BNF style"),while the restrictions involved in the canonical stage cannot - in general - be generated "algebraically", à la Montague say.However,the details needed for a complete description of such a "generative device",looking like Montague's "disambiguated languages", are unreasonably long and displaying them in full would rather obscure our main objective.

The free syntagmatic categories of LA may be chosen from a minimal family, containing all and only the following sets of words over A_{LA}:

$Term^{LA}$, the category of terms (in LA),

$Esent^{LA}$, the category of E-sentences (or E-formulas) in LA,

$Svar_n^{LA}$, the category of variable-strings of length n,n$\in\mathbb{N}$,in LA,

$Sterm_n^{LA}$, the category of term-strings of length n,n$\in\mathbb{N}$,in LA,

and

$Contx_n^{LA}$, the category of contexts of length n,n$\in\mathbb{N}$,in LA.

Of these,the first two are canonical,together with

$Constr^{LA}$, the category of canonical constructions in LA and

$Contx^{LA} = \bigsqcup_{n\in\mathbb{N}} Contx_n^{LA}$,the category of contexts in LA.

05.2.REMARK.

In the presentation adopted here (which does not concern ZA,the abstract version of Zucker's AUT-Pi;cf. ZUCKER 75-42,DE BRUIJN 77-51,ZANDLEVEN 77-48 and VAN DAALEN 80-73,Chapter VIII) it does not seem necessary to have, within the language,syntactic categories involving some notion of reduction (or any concept defined in terms of some notion of reduction,as e.g., "definitional equality" or "convertibility","reducibility" and "contraction"). It is,however,possible to adopt,from the very beginning,this point of view by admitting of one or more of the following categories of sentences (formulas) as primitive (both free and canonical) syntagmatic categories in LA:

$Qsent^{LA}$, the category of Q-sentences (or Q-formulas) in LA,

$Rsent^{LA}$, the category of R-sentences (or R-formulas) in LA,

and

$Csent^{LA}$, the category of C-sentences (or C-formulas) in LA.

As the latter are intended to "formalize" convertibility, reducibility and contraction in LA, one has to distinguish carefully between the underlying "extensionality type" (which can be either "beta-" or "beta-eta-" in the case of the languages studied below, while the distinctions necessary for the family ZA of abstract AUTOMATH languages are even more complex).
It will turn out that in all cases of concern below (not involving ZA say) Esent^{LA} is sufficient for all our purposes (and the remaining categories of sentences can be appropriately "simulated" or "defined" by a <u>détour via</u> the meta-language).
For some delicate points involving the use of (beta-)eta-convertibility in connection with correctness see, e.g., NEDERPELT 73-31, pp. 16, 71 and VAN DAALEN 80-73, Chapter VI. See also VAN DAALEN 80-73, <u>2.12.</u>, for a way out (avoiding the use of <u>Q</u>-formulas as primitives).

Now the only <u>text-category of</u> LA can be specified (schematically); it is

$\mathrm{Site}^{LA} = P_\omega(\mathrm{Constr}^{LA})$, the category of <u>sites in</u> LA,

(where, for any set $\underline{\underline{A}}$, $P_\omega(\underline{\underline{A}})$ is the set of <u>finite</u> subsets of $\underline{\underline{A}}$).

05.3. REMARK.

In the "reference" presentations of the languages in the AUTOMATH family one prefers the following terminology:

- "expressions" in LA — for elements of Term^{LA},
- "E-formulas" — for "<u>E</u>-sentences",
- "lines" (specifically, "primitive lines" or "PN-lines" and "definitional lines", resp.) — for "constructions"

and,

- "books" — for "sites" provided with a fixed <u>linear order</u> (a "book" is a "sequence of lines" and not merely a set, as it is the case here for our "sites").

Of course, "site", as used here, has nothing to do with the sites appearing in the theory of categories.
The motivation behind our use of sites may be recovered from our manipulation of a different (and more useful, both for meta-theoretical purposes and in practice) concept of order in an AUTOMATH "book" (cf. the discussion of the "reference order" below).

(iii) In order to give a "global" description of the correctness categories of LA we introduce some ad hoc set-theoretic notation and terminology.

05.4.NOTATION.TERMINOLOGY.

If R is an n-ary relation on $A_1,\dots,A_n$ $(n \geqslant 1)$ and $m \leqslant n$, then the m-th section of R is just R if m = n, else

$$Sec_m(R) = \{\langle a_1,\dots,a_m\rangle : \text{for some } a_i \in A_i,\ m+1 \leqslant i \leqslant n,\ \langle a_1,\dots,a_n\rangle \text{ is in } R\}.$$

If, moreover, $m < n$, and S is an m-ary relation on $A_1,\dots,A_m$ then the residual of S in R is the relation

$$R/S = \{\langle a_{m+1},\dots,a_n\rangle : \langle a_1,\dots,a_n\rangle \text{ is in R and } \langle a_1,\dots,a_m\rangle \text{ is in S}\}.$$

05.5.REMARK.

Now, clearly, if R is an n-ary relation then, the domain of R is the set

$$Dom(R) = Sec_{n-1}(R),$$

and the range of R is the set

$$Range(R) = R/Dom(R).$$

Now the correctness categories of (any abstract AUTOMATH language) LA are relations

$$Site_{\boxtimes}^{LA}, Constr_{\boxtimes}^{LA}, Contx_{\boxtimes}^{LA}, Esent_{\boxtimes}^{LA}, \text{ and } Term_{\boxtimes}^{LA}$$

and they can be viewed as set-theoretic "solutions" of the following "equalities" (and "inequalities", i.e., inclusions):

(1) $Site_{\boxtimes}^{LA} \sqsubseteq Site^{LA}$,

(2) $Constr_{\boxtimes}^{LA} \sqsubseteq Constr^{LA} \times Site^{LA}$, such that

$$Constr_{\boxtimes}^{LA}/Sec_1(Constr_{\boxtimes}^{LA}) = Site_{\boxtimes}^{LA} - \emptyset,$$

(3) $Contx_{\boxtimes}^{LA} \sqsubseteq Contx^{LA} \times Site^{LA}$, such that

$$Contx_{\boxtimes}^{LA}/Sec_1(Contx_{\boxtimes}^{LA}) = Site_{\boxtimes}^{LA},$$

(4) $Esent_{\boxtimes}^{LA} \sqsubseteq Esent^{LA} \times Contx^{LA} \times Site^{LA}$, such that

$$Esent_{\boxtimes}^{LA}/Sec_1(Esent_{\boxtimes}^{LA}) = Contx_{\boxtimes}^{LA},$$

(5) $Term_{\boxtimes}^{LA} \sqsubseteq Term^{LA} \times Contx^{LA} \times Site^{LA}$, such that

$$Term_{\boxtimes}^{LA}/Sec_1(Term_{\boxtimes}^{LA}) = Contx_{\boxtimes}^{LA}.$$

Of course, not any "solution" of the "system" above will be acceptable as a family of correctness categories for some abstract AUT-language.

1. Language definition: well-formedness.

We introduce first the "type-free" syntax of any abstract AUTOMATH language of concern below. Specifically: we define the syntactic categories of LA, as they appear in the free stage of generation and next build up, from these, a finite family of canonical (syntactic) categories (for LA).

10. Alphabets to use.

The alphabets we are going to use in these notes are singled out from the following denumerably infinite list of structured symbols. The categorization/ "classification" of these symbols into pairwise disjoint sub-alphabets is to be recovered from the tree-like indications present on the l.h.s. of the list displayed below.

We let $m,n \in \mathbb{N}$. To each constant in the list a non-negative integer, called arity (of the constant), is associated, indicating the "behaviour" of that constant qua syntactic function.

0	Variables: x_m	
1	(Structured) constants:	arity:
11	Functors:	
111	Floating constants:	
$1111_{n,m}$	p-constants: $p_{(n),m}$	m
$1112_{n,m}$	d-constants: $d_{(n),m}$	m
112	Constructors:	
1121	Term-constructors:	
11211_n	head-constructors ("instantiators"): δ_n	n
11212	abstractor: Λ	3
11213	applicator: $\ni$	2
1122	Sentence-constructors:	
11221	inhabitability relator: $\underline{E}$	2
1123	List-constructors:	
11231_n	variable sequencers: $\underline{s}_n$	n
11232_n	term sequencers: $\underline{S}_n$	n
11233	telescope constructor: $\underline{T}$	2
11234	"triad"-constructor: $\underline{D}$	3

12	Structural constants ("language constants"):	arity:
121	Universe symbols:	
1211	Super-type symbol: τ	0
1212	Proof-type symbol: π	0
122	Empty symbol ("nil"): $\square$	0.

10.1.COMMENT.

If the approach indicated in Remark 0.2. above is taken then one would also need the following extra <u>sentence</u> <u>constructors</u> (tree-like "classification" as earlier):

		arity:
11222	convertibility relator: $\underline{Q}$	2
11223	reducibility relator: $\underline{R}$	2
11224	contraction relator: $\underline{C}$	2.

These relators would serve as primitive constructors for the syntactic categories

$Qsent^{LA}$, formalizing convertibility sentences/formulas
$Rsent^{LA}$, formalizing reducibility sentences/formulas
$Csent^{LA}$, formalizing contraction sentences/formulas

in the correspondingly extended language LA.

From the list above we single out the alphabets to use as follows:

10.2.DEFINITION.

A contains the set of all structured symbols listed above.

$A_c = A - \{\pi\}$.

$A_p = A_c - \{\lambda, \ni\}$.

10.3.COMMENT.

A_p is the alphabet of PA (= Abstract Primitive AUTOMATH),
A_c is the alphabet of CA (= Abstract Classical AUTOMATH),
A is the alphabet of $Q^{-}A$,QA (= Abstract AUT-QE-NTI and Abstract AUT-QE).

In the sequel we shall somewhat simplify the exposition identifying A and A_c.

10.4.REMARK.

For theoretical purposes it is appropriate to identify A with A_c, leaving outside any considerations about the proof-type symbol π. The latter one is supposed to behave (up to a certain point) as τ does, so in order to restore the "official" formulation of the language from our description it will be enough to "duplicate" simply the syntax inasfar τ is concerned and to assume extra stipulations with τ replaced by π (*ceteribus paribus*). Actually, AUT-QE with a single universe symbol (τ) would be better suited for the formalization of *constructive* mathematics whereas the proof-type symbol would be necessary for "*classical*" texts (i.e., for books where the underlying logic is Classical Logic). See JUTTING 79-46, 4.1.2., VAN DAALEN 73-35, 3.2., ZUCKER 75-42, DE BRUIJN 80-72, VAN DAALEN 80-73, I.5.9. *sqq*., etc. A similar remark applies to Zucker's AUT-Pi (cf. ZUCKER *loc.cit*. and VAN DAALEN 80-73, VIII.).

The following meta-theoretic/syntactic notation will be used throughout in the sequel.

10.5.NOTATION.

Var stands for the set of *variables*, whereas Const^{LA} is the set of (structured) constants in LA (with superscript oft omitted).

We also write Pfl_n^{LA}, Dfl_n^{LA} (with n in $\mathbb{N}$) to denote the sets of n-ary floating p- resp. d-constants and take unions as follows:

$$\text{Fl}_n^{LA} = \text{Pfl}_n^{LA} \sqcup \text{Dfl}_n^{LA},$$

$$\text{Pfl}^{LA} = \bigsqcup_{n \in \mathbb{N}} \text{Pfl}_n^{LA}, \qquad \text{Dfl}^{LA} = \bigsqcup_{n \in \mathbb{N}} \text{Dfl}_n^{LA},$$

$$\text{Fl}^{LA} = \text{Pfl}^{LA} \sqcup \text{Dfl}^{LA}.$$

The superscript "LA" will be omitted whenever no confusions may arise.

Syntactic variables ("metavariables"):

v	for variables (in LA), with subscripts and successive primings to increase the list,
$\underline{p}$ or $\underline{p}_n$ (to specify the arity)	for floating p-constants (in LA),
$\underline{d}$ or $\underline{d}_n$	for floating d-constants, and
$\underline{c}$ or $\underline{c}_n$	for both.

The remaining (structured) constants will be used _autonymously_ in the meta-language and we won't adopt a special,very rigorous,notation for concatenation (see DE BRUIJN 77-49,78-61).Instead,readability is insured by some appropriate abuse of (meta-)language,often tacitly assumed.

10.6.TERMINOLOGY.NOTATION.

If A_{LA} is the alphabet of LA then Word(A_{LA}) or even Word(LA) denotes _the set of words over_ A_{LA}.We use X,Y,Z,...,etc.,as syntactic variables on Word(LA). _Epifunctions on_ Word(LA) are partial functions from Word(LA) to Word(LA).

10.7.REMARK.

As introduced below,some of the epifunctions appearing here can be defined in terms of other epifunctions.For our purposes it won't be necessary to have at hand a "minimal" basic list of such objects (for we do not aim at a complete formalization of the epitheory).

10.8.NOTATION.

Finally, $\equiv$ stands for syntactic identity on Word(LA) and we write " $\not\equiv$ " for its negation.

In set-theoretic contexts "=" has,of course,the usual set-theoretic meaning. Other set-notation is used as much as possible,in a standard way,and we did not find necessary to explain it in detail.

101. Further structure on (floating) constants.

A specific feature of the languages LA considered below consists of the presence of (what we called) floating constants (i.e., the structured symbols in Fl^{LA}). These are functors of a special kind and the way we intend to manipulate them in particular languages LA will - surprisingly - cause lots of (at least) theoretical difficulties. The discussion following here aims at eliminating the most typical ones from the very beginning.

The approach taken in this section is largely facultative and adopting the present point of view over any other possible set of theoretical decisions is certainly a matter of taste.

The main idea is to provide the sets Fl^{LA} (in each language LA) with a superimposed structure, which is, roughly speaking, an ordering. In the end, this amounts to a partition of Fl^{LA} into equivalence classes, called ranks, such that each rank is a (denumerably) infinite set of floating constants and there are also infinitely many ranks (the ranks will be, in fact, mapped one-one onto $\mathbb{N}$).

101.1. DEFINITION.

A ranking function for some alphabet A_{LA} is a map

$$\underline{\underline{rank}}: Fl^{LA} \longrightarrow \mathbb{N}$$

such that, for each (arity) $n, n \in \mathbb{N}$,

$$\underline{\underline{rank}}(Pfl_n^{LA}) = \mathbb{N}$$

and

$$\underline{\underline{rank}}(Dfl_n^{LA}) = \mathbb{N}.$$

101.2. CONVENTION.

Any alphabet A_{LA} is supposed to be given together with a fixed ranking function for it and to be "large" enough such as to have, for each arity $n, n \in \mathbb{N}$, and any given integer m, infinitely many p- resp. d-constants $\underline{c}_n$ with $\underline{\underline{rank}}(\underline{c}_n) = m$.

In detail, this runs as follows.

101.3. DEFINITION.

For each $n \in \mathbb{N}$, we let the appropriately restricted "kernels" of $\underline{\underline{rank}}$ be

$$Prank_n^m = \{\underline{p}_n : (\underline{p}_n \text{ is in } Pfl_n^{LA}) \& (\underline{\underline{rank}}(\underline{p}_n) = m)\} \qquad m \in \mathbb{N},$$

$$Drank_n^m = \{\underline{d}_n : (\underline{d}_n \text{ is in } Dfl_n^{LA}) \& (\underline{\underline{rank}}(\underline{d}_n) = m)\} \qquad m \in \mathbb{N},$$

101.4.REMARK.

Our previous convention on A_{LA} says that,for each arity n,$n \in \mathbb{N}$,and each $m \in \mathbb{N}$, the "ranks" $Prank^m_n$,$Drank^m_n$ can be mapped one-one onto $\mathbb{N}$.
The corresponding bijections allow to speak about the first,the second,..., the k-th p- resp. d-constant with arity n and rank m ($n,m \in \mathbb{N}$).

101.5.NOTATION.TERMINOLOGY.

$Prank^m = \bigsqcup_{n \in \mathbb{N}} Prank^m_n$, $\qquad Drank^m = \bigsqcup_{n \in \mathbb{N}} Drank^m_n$, $\quad (m,n \in \mathbb{N})$,

$Rank^m = Prank^m \sqcup Drank^m = \{ \underline{c} : (\underline{c} \text{ in } Fl^{LA}) \& (\underline{rank}(\underline{c}) = m) \} \qquad (m \in \mathbb{N})$.

For each non-negative integer m,we say that $Prank^m$,$Drank^m$,$Rank^m$ resp. is the p-rank m,the d-rank m or the rank m resp.(in LA).
Alternatively,if some floating constant $\underline{c}$ is in $Rank^m$ we say that $\underline{c}$ is of rank m ($m \in \mathbb{N}$) or (by abuse of language) that the rank of $\underline{c}$ is m.
In order to avoid superscripting we shall also write Rank(m) for $Rank^m$.

101.6.REMARK.

It is obvious that the ranking function for A_{LA} induces a partial order into Fl^{LA}.We shall find some use for this later on.

101.7.REMARK.

The point of view adopted here is,in the end,not very constructive.Still,we can make it to be so,by using explicitely Cantor's coding of the pairs in $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$.
In order to do this recall that,in 10. above,the p- and the d-constants were provided (separately) with some fixed lexical order (a linear order),and this was done for each arity n,$n \in \mathbb{N}$.
Indeed,we had p-constants $p_{(m),n}$ and d-constants $d_{(m),n}$,where $m \in \mathbb{N}$,in Pfl^{LA}_n, Dfl^{LA}_n resp. The intended meaning of the subscripts "(m)" is here that $p_{(m),n}$ is the (m+1)-th p-constant of arity n in A_{LA} and analogously for $d_{(m),n}$.
Now Cantor's coding is a (primitive) recursive function (a bijection)

$$\underline{pair}: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

with (primitive) recursive "projections" $\underline{left}$,$\underline{right}$ resp.Where $D_{\mathbb{N}}$ and $D_{\mathbb{N} \times \mathbb{N}}$ are the diagonals of $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ resp.,we get (by surjectivity) that the inverse,$\underline{pair}^-$,of $\underline{pair}$ exists,with

$$\underline{pair} \circ \underline{pair}^- = D_{\mathbb{N}} \quad \text{and} \quad \underline{pair}^- \circ \underline{pair} = D_{\mathbb{N} \times \mathbb{N}}$$

("o" denotes the usual composition of functions).
Let m be the lexical index of $p_{(m),n}$ resp. $d_{(m),n}$ in A_{LA}.Then $\underline{pair}^-(m)$ is

actually a pair, (m_1,m_2) say,with m_1,m_2 in $\mathbb{N}$.We may now consider m_1 as being <u>the rank of</u> the p- resp. d-constant with lexical index m (and arity n) and m_2 be <u>the lexical index of</u> the constant <u>within its rank</u> (here: either $Prank_n^m$ or $Drank_n^m$).For convenience,call the latter a <u>lexical rank-index</u>.
Taking,in particular,

$$\underline{\underline{pair}}(m_1,m_2) = ((m_1+m_2)x(m_1+m_2+1))/2 + m_2$$

it is easy to get explicit forms for <u>left</u>,<u>right</u> and <u>pair</u>$^-$ (see,e.g.,MAL'CEV 70).
So if <u>c</u> is a floating constant with lexical index r and arity n (r,n $\in \mathbb{N}$) then

$$\underline{\underline{rank}}^{\$}(\underline{c}) = \underline{\underline{left}}(\underline{\underline{pair}}^-(r))$$

defines (constructively) a ranking function for A_{LA} (for <u>rank</u>$^\$$ is obviously primitive recursive - and even Kalmár-elementary).
With the latter one we can repeat the partitioning of Fl^{LA} on the pattern sketched above and get the corresponding sets $Prank_n^{\$,m}$,$Drank_n^{\$,m}$,$Rank_n^{\$,m}$ (for all m,n in $\mathbb{N}$).
Moreover,each $\underline{p}_{(r),n}$ in $Prank_n^{\$,m}$ and each $d_{(r),n}$ in $Drank_n^{\$,m}$ will thereby get a <u>lexical rank-index within their</u> (constructive) <u>rank</u>,viz.

$$q = \underline{\underline{p\text{-}index}}_n^{\$}(p_{(r),n}) = \underline{\underline{right}}(\underline{\underline{pair}}^-(r)) \quad \text{and}$$

$$q = \underline{\underline{d\text{-}index}}_n^{\$}(d_{(r),n}) = \underline{\underline{right}}(\underline{\underline{pair}}^-(r))$$

and we may re-write the constants unambiguously as $p_{(q),n,m}$ and $d_{(q),n,m}$ resp.
Note that for any fixed non-negative integer q,a lexical rank-index q can be found,in each set Pfl_n and/or Dfl_n separately (for each n in $\mathbb{N}$),infinitely many times and,specifically,<u>only once in each rank</u> ($Prank_n^{\$,m}$ or $Drank_n^{\$,m}$).

101.8.REMARK.

If A_+ is any alphabet containing Fl^{LA} and $Word(A_+)$ is the set of words over A_+ we may also use the construction above in order to assign ranks to elements of $Word(A_+)$ or to subsets of $Word(A_+)$.Indeed,let X be in $Word(A_+)$.Then we may stipulate that the rank of X is -1 (say),if X does not contain letters from Fl^{LA} and that the rank of X is $\underline{\underline{max}}\,\{\underline{\underline{rank}}(\underline{c}):(\underline{c}$ occurs in X)&($\underline{c}$ in FL^{LA})$\}$.
Ranks -1 are,of course,arbitrary (we may leave the new ranking function undefined in those cases where X does not contain floating constants).
Something similar can be done for (finite) subsets of $Word(A_+)$.

101.9.COMMENT.

The need for something analogous to our ranking function above was noticed earlier by L.S. van Benthem Jutting and D.van Daalen,in connection with the theory of abbreviations (LSP) in AUTOMATH (see VAN DAALEN 80-73,III. and also,DE BRUIJN 73-30).In this context one speaks about the date of a defined constant (relative to some book),so the choice of a ranking function is a matter of local decision and has to be performed for each particular book separately (and anew).Also,in the latter case,the primitive constants (corresponding to our p-constants) must be dated (within "correct books"). Shifting the matter (in)to a very elementary syntactic level (as we did here) gives some technical advantages over the usual "reference"-treatement of the AUT-syntax,but also entails some unintended (and somewhat unpleasant) consequences at a later stage (definitional specifications are to be stipulated once forever for all elements of Dfl^{LA},etc.).

11. Variable-strings.

It will be convenient to start by defining strings of variables first, viz. the free (syntagmatic) categories $Svar_n^{LA}$, $n \in \mathbb{N}$. (We henceforth omit the superscripts "LA".).

These items do not differ in structure from language to language and, accurately, they cannot be generated in a context-free way.

11.1. DEFINITION.

For all n in $\mathbb{N}$, the sets $Svar_n$ are the least sets such that if $v_1,\dots,v_n$ are in Var and $v_i \neq v_j$ $(1 \leq i \neq j \leq n)$ then

$$\underline{s}_n \square v_1 \dots v_n$$

is in $Svar_n$.

An element of $Svar_n$ is a variable-string of length n.

11.2. REMARK.

Obviously, there is no point in referring to LA when speaking about variable-strings (this is also the case for variables, for we would want to keep Var the same for any LA under focus).

With the definition above one has a special notation for the empty string (of variables; this is just $\underline{s}_0\square$) and also one can distinguish straightforwardly - as intended - between a single variable v, taken in isolation, and the one-element string containing v (this is just $\underline{s}_1\square v$).

We have, by "sugaring" the syntax, the following.

11.3. NOTATION.

$$\boxtimes \ := \ \underline{s}_0\square,$$

$$\langle v_1,\dots,v_n \rangle \ := \ \underline{s}_n\square v_1\dots v_n, \qquad \text{for all } n \geq 1.$$

Analogously, we shall use the shorthand:

$$\bar{v}_n \ := \ \underline{s}_n\square v_1\dots v_n, \qquad \text{for all } n \geq 0,$$

omitting the subscript "n" whenever the length of the variable-string is known. The latter notation is used only when the missing information can be safely restored from the context.

If X is a variable-string we shall often write $\underline{\underline{lh}}(X)$ for the length of X.

12. Term-structures.

The LA-terms are the basic syntactic units in any LA and $Term^{LA}$, the set of terms in LA, is supposed to be specific for each LA.

For reasons of economy we identify hereafter $Term^{CA}$, $Term^{QA}$ and, in general, any set strictly containing $Term^{PA}$ will "collapse" into $Term^{CA}$. This amounts to the elimination of the proof-type symbol from the primitive syntax and, as noted earlier, the procedure is harmless for theoretic purposes. Indeed, the main differences induced by the presence of π seem to be rather semantic in nature and we are not concerned with this.

The syntax of LA-terms (at least in the free stage) will be not much more involved than that necessary in the construction of a combinatory reduction system, in the sense of KLOP 80 say, (this is a mixture of lambda-calculus and term-rewriting rules; cf. also HUET & OPPEN 80, etc.).

As for interpretation, the notion of an LA-term unifies two concepts which are distinct for languages with a separated type-structure, viz. in the latter case one has explicitely (object-)terms ("typed terms" say) and type-terms (accurately: "typing terms") while the type-terms can be generated by inductive definitions somewhat "beforehand", in complete independence from the former. Hence sometimes the labels: "languages with dependent type-structure" or "languages with generalized type-structure" - suggested by R.L.Constable and D.van Daalen resp., - for languages kindred to our LA's here. (For similar constructions see SCOTT 70, GIRARD 71,72, MARTIN-LÖF 71,72,75,75a,75b,79, BEESON 80, CONSTABLE 80, etc.).

121. Terms.

121.1. DEFINITION.

For LA as shown below, the set $Term^{LA}$ is the least set such that

(1) Any variable v is in $Term^{LA}$.

(2) The universe symbol τ is in $Term^{LA}$.

(3_n) If $\underline{c}$ is in Fl_n^{LA} and $a_1,\ldots,a_n$ are in $Term^{LA}$ then $\delta_n\underline{c}a_1\ldots a_n$ is in also in $Term^{LA}$. $(n \in \mathbb{N})$

(4) If a,b are in $Term^{LA}$ then so is $\ni ab$.

(5) If a,b are in $Term^{LA}$ and v is a variable then Λvab is in $Term^{LA}$.

In the above, if LA := PA then only clauses (1),(2) and (3_n), $n \in \mathbb{N}$, apply, for the applicator and the abstractor are not in the alphabet of PA.

121.2.DEFINITION.

Any element in $Term^{LA}$ is an LA-term or a term in LA.

121.3.COMMENT.

The "in" above should be unambiguous if we have at hand some appropriate book-keeping procedure for floating constants (in Fl^{LA}).The latter task can be somewhat simplified:if we choose to take into account the behaviour of LA-terms "modulo definitional expansion" then one needs to "store" only information concerning Pfl^{LA}.

121.4.REMARK.

Clearly,$Term^{PA}$ is a proper subset of $Term^{CA}$,while,for any other LA,one has by the simplifying convention on π above,that $Term^{CA} = Term^{LA}$.

We get,by "sugaring" again,the familiar "reference"-AUTOMATH notation.

121.5.NOTATION.

Hereafter a,b,c,d,e,f,g,... with sub- and/or superscripts (successive primings) will be used as syntactic variables on LA-terms (with LA duly specified).

Next,we set,for all $n \in \mathbb{N}$,

$$\underline{c}(a_1,\ldots,a_n) := \delta_n \underline{c} a_1 \ldots a_n \qquad \underline{c} \text{ in } Fl_n, a_i \text{ in } Term^{LA}, 0 \leq i \leq n,$$

$$\{a\}b := \ni ba \qquad a,b \text{ in } Term^{LA},$$

$$[v:a]b := \Lambda vab \qquad a,b \text{ in Term}, v \text{ in Var}.$$

We shall now largely diversify the epitheoretical landscape by supplying some useful descriptive terminology and introducing epifunctions to be employed later on.In general,our terminology will be closely related to standard ways of speaking in lambda calculus.

121.6.TERMINOLOGY.

(1) The head-terms are all and only the LA-terms of the form

$$a := \underline{c}(a_1,\ldots,a_n), \qquad n \in \mathbb{N},$$

with $\underline{c}$ and $a_1,\ldots,a_n$ as specified earlier;$\underline{c}$ is the head of a and the a_i's ($1 \leq i \leq n$) are the arms of a;the head and the arms of a are all and its only immediate quasi-components (hereafter: iqc's),while the arms of a are all and its only immediate components (henceforth: ic's).

The length of a is $\underline{\underline{lh}}(a) = n$.

Epifunctions: we let head and arm_i^n ($n \in \mathbb{N}, 0 \leq i \leq n$) be partial functions from Word(LA) to Word(LA) such that,

$$\text{head}(a) := \begin{cases} \underline{c}, & \text{if a is a head-term with head } \underline{c} \\ \text{undefined}, & \text{else} \end{cases}$$

$$\text{arm}_i^n(a) := \begin{cases} \mathbf{0}, & \text{if a is a head-term and } i = 0, \\ a_i, & \text{if } a := \underline{c}(a_1,\dots,a_n) \text{ and } \underline{lh}(a) > 0, 1 \leq i \leq n, \\ \text{undefined}, & \text{else.} \end{cases}$$

(2) The application-terms are all and the only LA-terms of the form

$$a := \{b_1\}b_2,$$

with b_1, b_2 in $Term^{LA}$; where b_1 is the argument-part of a and b_2 is the function-part of a; the argument- and the function-part of an application term a are all and the only iqc's (ic's) of a.

Epifunctions: the associated epifunctions are, in this case, arg and fun, with the following behaviour:

$$\text{arg}(a) := \begin{cases} b_1, & \text{if } a := \{b_1\}b_2, \\ \text{undefined}, & \text{else}, \end{cases}$$

$$\text{fun}(a) := \begin{cases} b_2, & \text{if } a := \{b_1\}b_2, \\ \text{undefined}, & \text{else.} \end{cases}$$

(3) The abstraction-terms are all and the only LA-terms of the form

$$a := [v:b_1]b_2,$$

with b_1, b_2 in $Term^{LA}$ and v in Var; here b_1 is the domain-part of a, while b_2 is its value-part and b_1, b_2 are all and the only ic's of a.
Finally, "unsugaring", $\Lambda v b_1$ is the abstraction prefix of a (but "sugaring" gives $[v:b_1]$, with brackets, for the latter) and b_1, b_2 and $\Lambda v b_1$ are all and the only icq's of a.

Epifunctions: the associated epifunctions are (see also below) dom and val, with:

$$\text{dom}(a) := \begin{cases} b_1, & \text{if } a := [v:b_1]b_2, \\ \text{undefined}, & \text{else} \end{cases}$$

and,analogously,

$$\underline{\underline{\underline{val}}}(a) \quad := \quad \begin{bmatrix} b_2, & \text{if } a := [v{:}b_1]b_2 \\ \text{undefined}, & \text{else.} \end{bmatrix}$$

121.7.REMARK.

(1) Any PA-term is either a variable or the universe symbol τ or a head-term.

(2) If LA is not PA,then any LA-term is either a PA-term or an application-term or an abstraction-term.

Indeed,this is the meaning of "least" in Definition 121.1.

121.8.DEFINITION.

A <u>subterm</u> (resp. <u>a quasi-subterm</u>) <u>of</u> an LA-term a is either

(1) an ic (an iqc) of a, or

(2) an ic (an iqc) of some subterm (quasi-subterm) of a,and

(3) nothing else is a subterm (quasi-subterm) of a,except by (1),(2) above.

121.9.TERMINOLOGY.

If a is an abstraction-term then any (quasi-)subterm b of

(i) $\underline{\underline{\underline{val}}}$(a) is <u>in the scope of</u> the abstraction prefix of a,

(ii) $\underline{\underline{\underline{dom}}}$(a) is <u>within the scope of</u> the abstraction prefix of a.

This way of speaking will be extended,by abuse of language,to sub-words of abstraction-terms that are not necessarily (quasi-)subterms of it.

121.10.TERMINOLOGY.

For any LA-term a,FV(a),the set of <u>free variables</u> of a,and BV(a),the set of <u>bound variables</u> of a,are supposed to be defined as usual.

We have,of course,BV(a) = $\emptyset$,for any PA-term a.

Further,if v is in Var,then v is said to be <u>free</u> (<u>bound</u>) <u>in</u> a if v is in FV(a),(resp. in BV(a),as expected),and v is <u>fresh for</u> a if it is not in FV(a) $\sqcup$ BV(a).

We shall also use "fresh for..." in a similar way,in connection with floating constants (a constant <u>c</u> is <u>fresh for</u> some LA-term a if,simply,<u>c</u> does not occur in a,as a quasi-subterm of it).

The operator of <u>simultaneous substitution on</u> LA-terms,for which we reserve the notation

$$a' \quad := \quad a[\![b_1,\ldots,b_n \; := \; v_1,\ldots,v_n]\!] \qquad\qquad n \in \mathbb{N},$$

(or even a' := $a[\![\overline{b} := \overline{v}]\!]$,where $\overline{b}$ and $\overline{v}$ stand for the corresponding sequences) is supposed to be defined in the familiar way.
This notation is straightforwardly relativized to the case of ordinary substitution (as in lambda calculus,say),when n = 1.

121.11.REMARK.

To avoid difficulties with alpha-conversion (see below) in languages with abstraction,it seems appropriate to introduce first partial operators of (simultaneous) substitution,next state the alpha-conversion rule and,finally, make the former totally defined on the appropriate set of terms,manipulating conveniently the alpha-matters (to handle this automatically some more details on alpha-conversion strategies are necessary,however).
This should avoid circularity,in the end,while defining substitution and alpha-conversion (the approach comes,essentially,from Church).
Alternatively,one has to cope with the alpha-matters in some other way (cf., e.g.,DE BRUIJN 72-29,78-55,STAPLES 79 and BARENDREGT 81,Appendix C: Variables).

122.Canonical terms and definitions.

The presence of floating constants in the alphabet(s) of LA induces some peculiarities that are absent from combinatory reduction systems (or term-rewriting systems) not containing such constants (i.e.,the so-called mechanism of "instantiation").

122.1.DEFINITION.

If $\overline{v} := \underline{s}_n \square v_1 \ldots v_n$ is in $Svar_n$ and $\underline{p}$ is in Pfl_n^{LA},$\underline{d}$ is in Dfl_n^{LA} $(n \in \mathbb{N})$ then the head-terms

$\underline{p}(\overline{v}) := \delta_n \underline{p} v_1 \ldots v_n$,

$\underline{d}(\overline{v}) := \delta_n \underline{d} v_1 \ldots v_n$,

resp. are said to be (canonical) p-terms (resp. d-terms) of length n and a canonical LA-term of length n is either a p-term or a d-term (of length n).

The canonical LA-terms are supposed to be characterized (in the metalanguage) by monadic ("epitheoretic") predicates called qualifiers.

122.2.TERMINOLOGY.

We distinguish between two kinds of qualifiers:

(1) improper qualifiers: we need only one of this form,viz. the "syntactic" predicate ISPRIM (read: "... is a primitive notion in/for LA"),and

(2) proper qualifiers: for each LA-term a,a proper qualifier $DEFEQUAL_a$ (read: "... is definitionally equal to a in/for LA").

122.3.NOTATION.

We shall write,everywhere in the sequel,for any LA-term a,

X ::: PRIM for ISPRIM(X)

and

X ::= a for $DEFEQUAL_a(X)$,

where X is a syntactic variable.

122.4.COMMENT.

The stipulation "in/for LA" is essential,if we want to recover information about some LA-term,which is not already "coded into" its syntactic form (there are CA-terms say that are apparently PA-terms whithout being actually so;we have to "restore their definitional history" in order to be able to put them at the due place,here PA vs CA).

122.5.NOTATION.

We let $Pterm_n^{LA}$,$Dterm_n^{LA}$ resp. be,for each $n \in \mathbb{N}$,the set of canonical p- resp. d-terms of length n and introduce sets $Pterm^{LA}$,$Dterm^{LA}$ by taking infinite unions in the obvious way ($Pterm^{LA}$ say is the union of all $Pterm_n^{LA}$,with n in $\mathbb{N}$). Similarly,$Cterm_n^{LA}$ is the set of canonical LA-terms of length n ($n \in \mathbb{N}$),while $Cterm^{LA}$ is the union of the latter sets,with n in $\mathbb{N}$.

122.6.DEFINITION.

(1) If $p(\bar{v})$ is in $Pterm_n^{LA}$ ($n \in \mathbb{N}$) then the epi-statement

$$\underline{p}(\bar{v}) \ ::: \ PRIM$$

is a pseudo-definition for LA with definiendum "$\underline{p}(\bar{v})$" and definiens PRIM. (This way of speaking involves some abuse of language but it will be useful later on.)

(2) Similarly,if $\underline{d}(\bar{v})$ is in $Dterm_n^{LA}$ ($n \in \mathbb{N}$) and a is in $Term^{LA}$ such that

(i) $FV(a) \sqsubseteq \{v_1,\dots,v_n\}$ and

(ii) $\underline{d}$ is fresh for a

then we say that the epitheoretic statement

$\underline{d}(\overline{v})$ ::= a

is a proper definition for LA with definiendum "$\underline{d}(\overline{v})$" and definiens "a".

122.7.COMMENT.

So far,any language LA would admit of several kinds of definitional ambiguity. We mention the most annoying cases:

(1) a given floating d-constant may be the definiendum of more than a single proper definition for LA,

(2) "circular definitions" are not avoided by the restriction concerning "freshness" in 122.6.

Certainly,(1) can be circumvented by adding a new stipulation concerning the relation which should subsist between d-constants and proper definitions in LA (this would be a "global" stipulation,concerning the "totality" of LA). The difficulty behind (2) was noticed by L.S. van Benthem Jutting (in conversation).Indeed,suppose $\underline{d}$ and $\underline{d}'$ are mutually distinct symbols in Dfl_0^{LA},for some LA (PA say).Then each of the following (proper) definitions are well-formed,and perfectly legitimate,when taken in isolation:

(2.1.) $\underline{d}$ ::= $\underline{d}'$

(2.2.) $\underline{d}'$ ::= $\underline{d}$.

This situation generalizes straightforwardly to any finite number of pairwise distinct d-constants of the same arity and the requirement of "freshness" is,obviously,satisfied,in each case separately.

We may still want to call such situations "well-formed" (one sometimes speaks about "good" vs "bad" definitions or "correct" vs "incorrect" definitions) and one could imagine that the due place to discuss such a distinguo is under the rubric "correctness".

This is,however,not the approach we shall actually take here.Rather,we have both good reasons and technical means to avoid such unpleasant features of our syntax from the very beginning.

As for reasons,we need not insist too much in arguing on the meaning of words.We simply adopt the point of view that a "bad / incorrect definition" is not a definition at all.

The technical means to eliminate "circular definitions" of the kind above are provided by putting the ranks (introduced in 101. above) at work.

We first need some auxiliary terminology.

Let,for further reference, $\underline{\underline{rank}}_{LA}$ be a fixed ranking function for LA (cf. 101.1.).

122.8.DEFINITION.

The rank of an LA-term (relative to $\underline{\underline{rank}}_{LA}$) is defined inductively as follows (if a is an LA-term,we write $\underline{\underline{rank}}^{+}_{LA}(a)$,for the rank of a in LA,relative to $\underline{\underline{rank}}_{LA}$,omitting often "LA"):

$$\underline{\underline{rank}}^{+}(a) = \begin{cases} -1, \text{if a does not contain floating constants as quasi-subterms,} \\ \underline{\underline{max}} \{ \underline{\underline{rank}}(\underline{c}) : (\underline{c} \text{ is in } Fl^{LA}) \& (\underline{c} \text{ is a qst of } a) \}. \end{cases}$$

The p- resp. the d-rank of an LA-term (relative to $\underline{\underline{rank}}_{LA}$) is defined analogously,replacing "Fl^{LA}" by "Pfl^{LA}" resp. "Dfl^{LA}" in the definition of $\underline{\underline{rank}}^{+}$ (notation: p-$\underline{\underline{rank}}^{+}_{LA}(a)$,d-$\underline{\underline{rank}}^{+}_{LA}(a)$,resp.).

122.9.REMARK.

It is easy to see that,for any LA-term a,one has

$$\underline{\underline{rank}}^{+}(a) = \underline{\underline{max}} \{ p\text{-}\underline{\underline{rank}}^{+}(a), d\text{-}\underline{\underline{rank}}^{+}(a) \}.$$

122.10.COMMENT.

For most of our purposes below only d-$\underline{\underline{rank}}^{+}$ will be relevant.However,using $\underline{\underline{rank}}^{+}$ instead does not introduce sensible complications.
For any LA-term a,$\underline{\underline{rank}}^{+}(a)$ is a measure of complexity for a.Accurately,this is completely relevant only for PA-terms,for $\underline{\underline{rank}}^{+}(a)$ "ignores" the number (say) of occurrences of abstractors and applicators in a.
In order to get a correct estimation of the syntactical complexity of an LA-term (whether it is a PA-term or not) we would likely need counting - some way or another - the occurrences of the latter constructors in the term concerned.
A simple solution seems to be as follows: let a be an LA-term and define ad hoc

$-$\underline{\underline{abs}}$(a) := the number of abstractors occurring in a,

$-$\underline{\underline{app}}$(a) := the number of applicators occurring in a.

Now,with p = $-$\underline{\underline{abs}}$(a) and q = $-$\underline{\underline{app}}$(a) and where r = $\underline{\underline{rank}}^{+}(a)$ one may define the syntactical complexity of an LA-term a as being

$$\underline{\underline{c\text{-}synt}}(a) = 5^{p} + 3^{q} + 2^{r+2}.$$

(This won't be very useful,however.)
On the other hand,d-rank (a) is,for any LA-term a,a faithful measure of definitional complexity for a.Since $\underline{\underline{rank}}_{LA}$ is supposed to be fixed for each LA,from

the very beginning, this "measure of complexity" is also "absolute" for any LA-term a (cf. 101. above).

We come now back to our main task, viz. we want to give an estimation of the "freshness" of a floating constant for a given LA-term.
The proposal following below is somewhat arbitrary but it will work satisfactorily in all cases of concern.

122.11. DEFINITION.

Let a be an LA-term with $\mathrm{rank}^+(a) = n, n \geqslant -1$. Then a floating constant c is said to be

(1) rank-fresh for a if $\mathrm{rank}(c) > \mathrm{rank}^+(a)$

and

(2) minimally fresh for a if $\mathrm{rank}(c) = \mathrm{rank}^+(a) + 1$.

122.12. REMARK.

Clearly, for all LA-terms a and all c in Fl^{LA},

(1) if c is rank-fresh for a then c is fresh for a

and

(2) if c is minimally fresh for a then it is also rank-fresh for a,

but the converses of these implications are, in general, not true.

122.13. COMMENT.

We did not restrict the definition of rank-freshness and minimal freshness to d-constants for reasons which will appear in the statement of the correctness rules of LA's below (see 2.). But for the purposes of this section the restricted definitions would work.

Now we can state a "global" condition of definitional disambiguation for any language LA of concern here.

122.14. CONVENTION.

The languages LA are assumed to be definitionally unambiguous in the following sense:

(1) each floating p-constant in Pfl^{LA} has a pseudo-definition for LA,

(2) for each floating d-constant d in Dfl_n^{LA} ($n \in \mathbb{N}$), there is exactly one proper definition for LA, having "$d(v_1,\dots,v_n)$" as definiendum, and

(3) for each proper definition D for LA, the head of the definiendum of D is rank-fresh for the definiens of D.

122.15.COMMENT.

It is easy to see that the difficulties noted in 122.7. above are straightforwardly circumvented by adopting Convention 122.14.In particular,one can adopt even a less liberal restriction in 122.7.(3),taking "minimally fresh" instead of "rank fresh".

We are completing our discussion of definitions by some more terminological improvements.

122.16.TERMINOLOGY.

A definition(al specification) for LA is either a pseudo-definition for LA or a proper definition for LA.

We say that a definition specifies its definiendum or that it is a definitional specification of its definiendum for LA.

The meta-linguistic phrase "where" (called,for historical reasons,Landin abstractor; cf.,e.g.,LANDIN 66;it actually acts as an abstractor within the epi-theory) will be used as an epi-theoretic operator on definitional specifications for LA's and it will contribute to the formation of epi(-theoretic)-phrases called "where-clauses" (for LA).

122.17.DEFINITION.

A where-clause for LA is an epi(-theoretic)-phrase

where ' definitional specification '

and the definitional specification following the Landin abstractor "where" is the supporting definition of the where-clause (or its body).

122.18.TERMINOLOGY.

We also say that a where-clause has a specified canonical LA-term and note that the latter is the definiendum of its supporting definition.The head of the specified canonical term is the specified constant of the corresponding where-clause.

122.19.CONVENTION.

We will often omit where-clauses having a pseudo-definition as supporting definition (body).

123. Combinatory reduction systems.

With each language LA a special relational structure on $Term^{LA}$ will be associated, called (after KLOP 80) "combinatory reduction system" (crs, for short).

These structures arose by generalizing both the lambda calculi and the term rewriting systems (of which Curry's - and Rosser's - "combinatory logics" are particular cases; cf. CURRY et al. 58,72; HINDLEY et al. 72; REZUS 81) in view of obtaining a general setting for the Church-Rosser Theorem (BARENDREGT 81, KLOP 80).

The crs's of concern here are either regular in the sense of KLOP 80 (cf. Klop's Chapter II, for definitions) or are eta-extensions of regular structures. That is: in the end, the Church-Rosser property is, in general, insured by "global" arguments for notions of reduction not involving "eta-reduction".

This is not the case for relational structures associated to the abstract AUT-language ZA (or Zucker's AUT-Pi; the first proof of the fact that some "free" subsystem of AUT-Pi does not satisfy the Church-Rosser property is - implicitely - due to J.W.Klop; see KLOP 79; the fragment concerned, lambda calculus with surjective pairing, is, however, consistent by a model-theoretic argument, using Scott-like models; cf. SCOTT 80; a syntactic proof of consistency for this fragment of AUT-Pi is planned for DE VRIJER 8[illegible]).

Church-Rosser also fails, in a type-free setting, for crs's associated to an (abstract) AUT-language which involve both beta- and eta-reduction (and the problem arises only when the latter notion is present; this was first noticed in NEDERPELT 73-31, page 71). Specifically, the failure is caused by the presence of (AUTOMATH-) labels, i.e., the "types" that are domain-parts.

Fortunately, the Church-Rosser property holds for the reducibility relation restricted to "correct" AUT-expressions (if AUT is not AUT-Pi) and the result transfers easily to the corresponding abstract versions (cf. VAN DAALEN 80-73, Chapter VI, for details and the due proofs).

Somewhat simplifying, the crs's to be discussed below are either ("labelled") term rewriting systems or ("labelled") lambda calculi with additional term rewriting rules.

We introduce first various notions of reduction, following BARENDREGT 81, Chapter 3 (or VAN DAALEN 80-73; cf. with the "elementary reductions" of the latter, loc. cit. II.3.). These are, simply, binary relations on $Term^{LA}$.

The relational structures to be considered here arise from (combinations of) these notions, taking the appropriate closures on $Term^{LA}$.

So, the description of a crs is, prima facie, a second-order description. Anyway, first-order analogues won't be very profitable for our purposes.

Some closures (of the due notion of reduction) will be taken as basic in the definition of the coresponding crs. These are binary relations on (appropriate sets) $Term^{LA}$, called resp. contraction, reducibility and convertibility (or definitional equality) on $Term^{LA}$.

However, unlike in BARENDREGT 81, these relations will be viewed as belonging to the epi-theory (or, if one prefers, to the meta-language) of the associated LA's rather than being incorporated in the "object language" itself.

So we consider structures on $Term^{LA}$ and not formal systems. The latter approach follows, mutatis mutandis, REZUS 81.

This agrees, in the end, with the actual practice of "reference"-AUTOMATH, where the matters concerning reducibility or (the) definitional equality of terms (there: "expressions") or their "termination properties" are handled via some checking device (implemented and processed by a computer), somewhat "physically" outside any "book". Still, this is not the approach usually taken in presenting the language definition of "reference"-AUTOMATH.

So any considerations concerning the definitional equality of LA-terms will be shifted (in)to the epi-theory/meta-language of the corresponding LA.

Some (non-trivial) technical tricks will be needed in order to do this for all abstract AUT-languages (but ZA; in the latter case one has to work exclusively with some formal system, for reduction cannot be separated from correctness considerations). The main solutions were suggested by D. van Daalen (in VAN DAALEN 80-73, Chapter V., 2.12. and in conversation.).

We need first some preliminary notions.

123.1. DEFINITION.

A binary relation $\underline{\underline{R}}$ on $Term^{LA}$ (infix notation) is

(1) monotone relative to δ_n ($n \in \mathbb{N}$) if, for all a_i, a_i' in $Term^{LA}$ and all $\underline{c}$ in Fl_n^{LA} ($0 \leq i \leq n$),

$$a_i \;\underline{\underline{R}}\; a_i' \implies \delta_{n-1}\underline{c}a_1 \ldots a_i \ldots a_n \;\underline{\underline{R}}\; \delta_{n-1}\underline{c}a_1 \ldots a_i' \ldots a_n;$$

(2) monotone relative to $\ni$, if, for all a_1, a_2, b_1, b_2 in the set $Term^{LA}$ (where LA is not PA) one has both

$a_1 \underline{\underline{R}} b_1 \quad \Longrightarrow \quad \exists a_1 a_2 \underline{\underline{R}} \exists b_1 a_2$

and

$a_2 \underline{\underline{R}} b_2 \quad \Longrightarrow \quad \exists a_1 a_2 \underline{\underline{R}} \exists a_1 b_2$;

(3) monotone relative to Λ ,if,for all a_1,a_2,b_1,b_2 in $Term^{LA}$ and all variables v (where LA is not PA),one has both

$a_1 \underline{\underline{R}} b_1 \quad \Longrightarrow \quad \Lambda v a_1 a_2 \underline{\underline{R}} \Lambda v b_1 a_2$

and

$a_2 \underline{\underline{R}} b_2 \quad \Longrightarrow \quad \Lambda v a_1 a_2 \underline{\underline{R}} \Lambda v a_1 b_2$.

123.2.DEFINITION.

A binary relation $\underline{\underline{R}}$ on $Term^{LA}$ is monotone in LA if Op^{LA} is the set of constructors in the alphabet of LA and $\underline{\underline{R}}$ is monotone relative to each $\underline{o}$ in Op^{LA}.

123.3.DEFINITION.

Let $\underline{\underline{R}}$ be a binary relation on $Term^{LA}$ (LA,as ever).Then the monotone closure of $\underline{\underline{R}}$ in LA is the least relation $\underline{\underline{R}}^{\boxtimes}$ such that $\underline{\underline{R}}^{\boxtimes}$ contains $\underline{\underline{R}}$ and is monotone in LA.

123.4.REMARK.NOTATION.

One defines,as expected,the reflexive and transitive closure of a binary relation $\underline{\underline{R}}$ on $Term^{LA}$ (in LA).

Notation: if $\underline{\underline{R}}$ is a binary relation on $Term^{LA}$ then $\underline{\underline{R}}^{\wedge}$ denotes the reflexive and transitive closure of R in LA.

For languages with abstraction we need some appropriate re-lettering device for bound variables ("dummies").It is appropriate to cope with this (the "alpha matters") first in one way or another.

The main reason behind the use of alpha-convertibility in some lambda calculus consist in that one wants to have at hand - within the calculus - an expedient tool of disambiguating terms containing abstraction terms as subterms.

Actually,in the ("reference"-)AUT-languages the "alpha matters" play no explicit rôle,for the underlying lambda calculus becomes,after implementation,a "nameless lambda calculus" (cf. DE BRUIJN 72-29,78-55;BARENDREGT 81,Appendix C),where bound variables are not specified any more,but handled by an appropriate system of references which eliminates any possible ambiguity.

For present purposes,it won't be neccessary to insist with formal details concerning alpha-convertibility.Rather,we shall adopt a set of practical conventions which will allow us to work with LA-terms in the "naive way". Specifically,where alpha-congruent LA-terms are defined in the usual way (cf. mutatis mutandis,BARENDREGT 81,2.1.11.),we agree to adopt Conventions 2.1.12. and 2.1.13. in BARENDREGT 81,say (any two alpha-congruent LA-terms are identified, and any LA-term is "basically disambiguated" in the sense that no variable occurring in it is both a free and a bound variable in that LA-term).
In particular,this will cause no specific problem when passing from PA to CA,QA, etc.,for we won't work with "equivalence classes" modulo alpha-conversion,but rather with "representatives" in such classes.

Now we can turn back to our main task.

123.5.DEFINITION.

A notion of reduction in LA is a binary relation on $Term^{LA}$.

123.6.NOTATION.

If $\underline{\underline{R}}_1,\underline{\underline{R}}_2$ are notions of reduction in LA then $\underline{\underline{R}}_1\underline{\underline{R}}_2$ stands for their union.

123.7.COMMENT.

The notions of reduction of concern here will be given via graphs of partial recursive functions on $Term^{LA}$ (for appropriate LA's).So it is implicitely assumed that Pfl_n^{LA},Dfl_n^{LA} ($n \in \mathbb{N}$) are specified recursively,as actually intended in the "reference"-versions of AUTOMATH.

123.8.NOTATION.

We display the actual definitions by writing

a $\underline{\underline{R}}$ b (such that) "... a ... b ... "

instead of using the "graph-notation" for $\underline{\underline{R}}$.

This runs as follows:

123.9.DEFINITION.("Delta-reduction".)

Where $\overline{v} := \{ v_1,\ldots,v_n \}$ is in $Svar_n$,$a_1,\ldots,a_n$ are in $Term^{LA}$,$\underline{d}$ is in Dfl_n ($n \in \mathbb{N}$) and "where $\underline{d}(\overline{v})$ ::= b" is a where-clause for LA,

$$\underline{d}(a_1,\ldots,a_n) \ \underline{\underline{delta}} \ b[\![\overline{a} := \overline{v}]\!] ,$$

with $\underline{d}(\overline{v}) := \underline{d}(v_1,\ldots,v_n)$,as expected.

In this case,"$\underline{d}(a_1,\ldots,a_n)$" is a delta-redex and "$b[\![\overline{a} := \overline{v}]\!]$" is its contractum.

123.10.DEFINITION.("Beta-reduction".)

For all a_1,a_2,b in $Term^{LA}$, all variables v (where LA is not PA),

$$\{b\}[v:a_1]a_2 \ \underline{\underline{beta}} \ a_2[\![b := v]\!].$$

123.11.DEFINITION.("Eta-reduction".)

For all a_1,a_2 in $Term^{LA}$, all variables v (where LA is not PA),

$$[v:a_1]\{v\}a_2 \ \underline{\underline{eta}} \ a_2,$$

provided v is not in $FV(a_2)$.

123.12.REMARK.TERMINOLOGY.

In the latter two cases redexes and contracta are defined in the usual way. Hence we should be able to define - as expected - delta- ,beta-,eta-normal forms as well as combinations of the above.(An LA-term is in delta-normal form if no subterm of it is a delta-redex,etc.)

123.13.NOTATION.

For convenience we shall denote the notions of reduction introduced above by the appropriate (lower-case) bold-face Greek letters.So $\underline{\underline{\delta}}$:= $\underline{\underline{delta}}$, $\underline{\underline{\beta}}$:= $\underline{\underline{beta}}$, $\underline{\underline{\eta}}$:= $\underline{\underline{eta}}$ and we shall take unions as stipulated above (123.6.).

123.14.DEFINITION.

Let $\underline{\underline{R}}$ be some notion of reduction in LA.Then

(1) the corresponding relation of contraction in LA is the monotone closure of $\underline{\underline{R}}$ in LA (that is: $\underline{\underline{R}}^{\boxtimes}$).Notation: $\underline{\underline{contr}}_{LA}$.

(2) Reducibility in LA is the reflexive and transitive closure of contraction in LA.Notation: $\underline{\underline{red}}_{LA}$.

(3) Convertibility (or definitional equality) in LA is the least equivalence containing $\underline{\underline{contr}}_{LA}$ (on $Term^{LA}$).Notation: $\underline{\underline{conv}}_{LA}$.

123.15.NOTATION.

We use in the sequel:

(1) delta-contraction : $\underline{\underline{contr}}_{\delta} := (\underline{\underline{\delta}})^{\boxtimes}$,

delta-reducibility: $\underline{\underline{red}}_{\delta}$

delta-convertibility: $\underline{\underline{conv}}_{\delta}$

only in connection with PA.

In the remaining cases,each LA is supposed to be formulated with two distinct "extensionality-types" (terminology from J.P.Seldin and H.B.Curry),viz. the "beta-type" and/or the "beta-eta-type".This gives abstract AUT-languages β-LA and $\beta\eta$-LA, formulated respectively, with:

(2) beta-delta-contraction: $\underline{\underline{contr}}_{\beta\delta} := (\underline{\underline{\beta\delta}})^{\boxtimes}$,
beta-delta-reducibility: $\underline{\underline{red}}_{\beta\delta}$,
beta-delta-convertibility: $\underline{\underline{conv}}_{\beta\delta}$

and

(3) beta-eta-delta-contraction: $\underline{\underline{contr}}_{\beta\eta\delta} := (\underline{\underline{\beta\eta\delta}})^{\boxtimes}$,
beta-eta-delta-reducibility: $\underline{\underline{red}}_{\beta\eta\delta}$,
beta-eta-delta-convertibility: $\underline{\underline{conv}}_{\beta\eta\delta}$.

We shall however keep using the labels LA ambiguously as long as the distinction between $\beta\delta$- and $\beta\eta\delta$-reducibility/convertibility plays no rôle in the description of the underlying syntax.

123.16.DEFINITION.

A reduction system (or an LA-calculus or even a crs) on LA is a (relational) structure

$$\underline{\underline{LA}} = \langle \mathrm{Term}^{LA}; \underline{\underline{contr}}_{LA}, \underline{\underline{red}}_{LA}, \underline{\underline{conv}}_{LA} \rangle,$$

where

(1) Term^{LA} is the set of terms in LA and
(2) $\underline{\underline{contr}}_{LA}, \underline{\underline{red}}_{LA}, \underline{\underline{conv}}_{LA}$ are resp. contraction, reducibility and convertibility in LA.

A reduction system on LA is extensional if $\underline{\underline{contr}}_{LA}$ contains $\underline{\underline{eta}}$, otherwise it is non-extensional. (This terminology will be also transferred to the corresponding language LA.)

123.17.COMMENT.

It appears that the behaviour of the extensional abstract AUT-languages (and associated crs's) diverges in several important respects from that of their non-extensional variants (due to the presence of "labels", i.e., "types" that are domain-parts). This is not so for the corresponding pure ("un-labelled" or "type-free") lambda calculi, even if delta-reduction is present.
Hence we have to examine the extensional case separately.

123.18.THEOREM. (The Church-Rosser Property for LA := PA, β-CA, β-QA, β-$\bar{Q}$A).

For all LA-terms a,b,c and $\underline{\underline{red}}_{LA}$ in the reduction system on LA
if a $\underline{\underline{red}}_{LA}$ b and a $\underline{\underline{red}}_{LA}$ c then an LA-term d can be found such that
b $\underline{\underline{red}}_{LA}$ d and c $\underline{\underline{red}}_{LA}$ d.

Proof. By any method involving (explicitely or not) "residuals". See KLOP 80, BARENDREGT 81, VAN DAALEN 80-73, REZUS 81. $\boxtimes$

It can be shown easily that the preceeding theorem is, in fact, equivalent to the following "confluence property":

123.19. (The Church-Rosser Theorem for LA := PA, β-CA, β-QA, β-Q^{-}A.)

For all LA-terms a,b, where $\underline{red}_{LA}$, $\underline{conv}_{LA}$ are as expected,

if a $\underline{conv}_{LA}$ b then, for some LA-term c, a $\underline{red}_{LA}$ c and b $\underline{red}_{LA}$ c.

Proof. Trivial, using 123.18.⊠

The latter fact gives the following reassuring result (not very useful, however, in the present context).

123.20. THEOREM. (Consistency for LA := PA, β-CA, etc.)

Convertibility in LA is a proper subset of the cartesian product $Term^{LA}$ x $Term^{LA}$.

(That is: not any two LA-terms can be identified via $\underline{conv}_{LA}$ in $\underline{LA}$.)

Proof. Easy, by reductio, from 123.18.⊠

123.21. COMMENT.

Theorems 123.18., 123.19. and 123.20. hold also for the extensional reduction $\underline{red}_{\beta\eta\delta}$ on $Term^{LA}$, provided one ignores "labels" (i.e., "types" that are domain-parts). More precisely: erase, from any abstraction term a in $Term^{LA}$, $\underline{dom}$(a), i.e., consider the abstraction terms without domain-parts. The resulting set of (LA-)terms is "type-free" and it is easy to show (by standard methods) that the corresponding relation $\underline{red}_{\beta\eta\delta}$ is actually Church-Rosser (in the sense of 123.18.) on this set.

Even in absence of 123.18., Theorem 123.20. holds for the "labelled" calculi βη-$\underline{LA}$ (where, of course, LA is not PA) by a model-theoretic argument. Alternatively, one can use the Church-Rosser property for the corresponding "type-free" language/structure, realizing that one cannot "identify" more LA-terms by "labelling". (See KLOP 80, BARENDREGT 81 for a detailed explanation of the "labelling techniques".)

Somewhat more useful a consequence of the Church-Rosser Theorem for PA and β-LA's is the Unicity of Normal Forms (UN; cf. KLOP 80, etc.).

123.22.DEFINITION.

An LA-term a has a delta- (beta-delta-,beta-eta-delta-)normal form if

(1) a $\underline{red}_{LA}$ b

(2) b is in delta- (beta-delta-,beta-eta-delta-)normal form.

(Alternative terminology: a is weakly normalizable in LA,a is LA-normalizable.)

Now the UN-property can be stated as follows:

123.23.THEOREM.

(1) If a PA-term is PA-normalizable then its delta-normal form is unique.

(2) If an LA-term a is β-LA-normalizable then its beta-delta-normal form is unique (up to alpha-conversion;but see the conventions on alpha above).

Proof.By 123.18.⊠

23.24.COMMENT.

Theorem 123.23. does not hold for extensional structures (associated to abstract AUT-languages).

Indeed,consider the following counter-example (which goes back to NEDERPELT 73-31,and also falsifies Theorems 123.18.,123.19.).

Let p,p' be floating constants in Pfl_0^{LA} (0-ary).They are LA-terms in $\beta\eta\delta$-normal form , in some $\beta\eta$-LA.(As a "reference"-AUT example take,e.g.,nat for p and real for p',denoting resp. the type of naturals and the type of reals,qua primitive notions.)

Consider then the LA-terms

a := [v:p] v and a' := [v:p'] v.

One has,in any $\beta\eta$-LA,that

a $\underline{conv}_{\beta\eta d}$ a',

for,with

b := [v:p] {v} [v':p'] v'

we can find immediately that

b $\underline{red}_{\beta}$ a and b $\underline{red}_{\eta}$ a'

and $\underline{red}_{\beta\eta}$ is a subset of $\underline{red}_{\beta\eta\delta}$ and $\underline{conv}_{\beta\eta\delta}$ as well.

But a and a' are in $\beta\eta\delta$-normal form.So b has two (alpha-)distinct $\beta\eta\delta$-normal forms.

Of course,erasing domain-parts - i.e.,dom(a) and dom(a') - restores UN in a "type-free" setting.

By the same token,the Church Rosser Property fails for $\beta\eta$-LA's,for - in the example above - a and a',being both "reducts of b" in $\beta\eta$-LA,have no "common reduct".

13. Term-strings.

For reasons which will appear below we need a more liberal notion of a string. It is reasonable to employ constructors different from the $\underline{s}_n$'s in 11. when defining this notion; that is: we have to introduce a new (free) syntagmatic category in LA.

13.1. DEFINITION.

The sets Sterm_n^{LA} ($n \in \mathbb{N}$) are the least sets such that

if $a_1,\dots,a_n$ are in Term^{LA} then $\underline{S}_n \square a_1 \dots a_n$ is in Sterm_n^{LA}.

13.2. DEFINITION.

An element in Sterm_n^{LA} is a term-string of length n (in LA), $n \in \mathbb{N}$.

13.3. NOTATION.

As earlier, for Svar_n's we write

$$\boxtimes \;:=\; \underline{S}_0 \square$$

$$\langle a_1,\dots,a_n \rangle \;:=\; \underline{S}_n \square a_1 \dots a_n \qquad (n \geq 1).$$

The length of a term-string will be denoted by $\underline{\underline{lh}}(\bar{a}_n) = n$, where

$$\bar{a}_n \;:=\; \underline{S}_n \square a_1 \dots a_n \qquad (n \geq 0),$$

(we omit subscripts "n" whenever the length of the term-string $\bar{a}_n$ is known).

13.4. COMMENT.

We have "sugared" the empty strings $\underline{s}_0\square$ and $\underline{S}_0\square$ in the same way (though they are "syntactically distinct within the theory"). This is not too important, for they will be distinguished (again) positionally, in more involved syntactic units. But if the reader does not like confusing empty strings (!) he can leave "$\boxtimes$" "unsugared" in both cases.

13.5. NOTATION. TERMINOLOGY.

As epifunctions, on both Svar_n and Sterm_n^{LA}, $n \in \mathbb{N}$, we shall use the following partial functions $\underline{\underline{elt}}_i^n$ ($0 \leq i \leq n$), defined by

$$\underline{\underline{elt}}_i^n(X) \;:=\; \begin{cases} \square, & \text{if } X := \boxtimes \text{ (i.e., } X := \underline{s}_0\square \text{ or } \underline{S}_n\square) \\ v_i \text{ or } a_i, & \text{if } X := \langle v_1,\dots,v_n \rangle \text{, or} \\ & \quad X := \langle a_1,\dots,a_n \rangle,\ 1 \leq i \leq n, \\ \text{undefined, else.} & \end{cases}$$

Also, we set, for all $n \in \mathbb{N}$, $\underline{\underline{lastelt}} := \underline{\underline{elt}}_n^n$ (understanding that $\underline{\underline{lastelt}}(\boxtimes) \equiv \Box$). The latter gives the "last element" of a string whenever its input is of the due kind.

Now we are able to introduce two more epifunctions from LA-terms to strings. These are $\underline{\underline{vtail}}$ and $\underline{\underline{tail}}$, defined by

$$\underline{\underline{vtail}}(X) := \begin{cases} \boxtimes, & \text{if X is a canonical term}, \underline{\underline{lh}}(X) = 0 \\ \langle v_1,\ldots,v_n \rangle, & \text{if X is a canonical term}, X := a, \text{ and } a \equiv \delta_{n} c v_1 \ldots v_n \\ \text{undefined}, & \text{else.} \end{cases}$$

$$\underline{\underline{tail}}(X) := \begin{cases} \boxtimes, & \text{if } X := \delta_{n} c \text{ (a head-term}, \underline{\underline{lh}}(X) = 0) \\ \langle a_1,\ldots,a_n \rangle, & \text{if } X := \delta_{n} c a_1 \ldots a_n, \underline{\underline{lh}}(X) = n, n \geqslant 1. \\ \text{undefined}, & \text{else.} \end{cases}$$

That is: where $Hterm^{LA}$ is the set of head-terms in LA, $\underline{\underline{tail}}$ is a total function from $Hterm^{LA}$ to $Sterm^{LA} = \bigsqcup_{n \in \mathbb{N}} Sterm_n^{LA}$, whereas $\underline{\underline{vtail}}$ is total from canonical terms to variable-strings.

13.6. COMMENT.

Note that for v_i in Var $(1 \leqslant i \leqslant n)$, whenever $\bar{v} := \langle v_1,\ldots,v_n \rangle$ is in $Svar_n$, we have $v_i \neq v_j$ for $1 \leqslant i \neq j \leqslant n$, while if $\bar{v}$ is in $Sterm^{LA}$ this is not required. The distinction between $\underline{s}_0 \Box v_1 \ldots v_n$ and $\underline{S}_0 \Box v_1 \ldots v_n$ will be convenient later (although it looks somewhat strange).

14. E-sentences.

14.1. DEFINITION.
The sets $Esent^{LA}$ are the least sets such that, if a,b are in $Term^{LA}$ then $\underline{E}ab$ is in $Esent^{LA}$.
The elements of $Esent^{LA}$ are called E-sentences or E-formulas in LA (LA := PA, CA, QA, $\bar{Q}$A, etc.)

By "sugaring" the syntax we get some familiar notation.

14.2. NOTATION.
We shall write in the sequel,

$$(a:b)$$

for

$$\underline{E}ab$$

but outermost parentheses will be omitted whenever no confusions may arise.
Lower-case Greek letters ϕ, ψ, possibly with sub- and/or superscripts will be used as syntactic variables for E-sentences.

14.3. DEFINITION.
If $\phi := \underline{E}ab$ is an E-sentence in LA then a is the subject of ϕ and b is its predicate.

14.4. DEFINITION.
An assumption is an E-sentence ϕ whose subject is a variable.

14.5. DEFINITION.
A (canonical) p-sentence (resp. d-sentence) is an E-sentence ϕ whose subject is a canonical p-term (resp. a canonical d-term).
A canonical E-sentence is either a canonical p-sentence or a canonical d-sentence.

14.6. REMARK.
Of course, "in LA" should be supplied everywhere in the above.

As expected, we would want to speak about the "components" of an E-sentence without mentioning them explicitely. This will be done by using appropriate epifunctions, sub and pred.

14.7.NOTATION.

If Φ := Eab (Φ is an E-sentence in LA) then

$$\mathrm{sub}(\Phi) \equiv a \qquad \text{and} \qquad \mathrm{pred}(\Phi) \equiv b,$$

(else sub and pred are undefined).

Now we may introduce, by convention, a new epifunction from LA-terms to E-sentences in LA (where LA is not PA), abs say.

14.8.NOTATION.

For all X in Word(LA),

$$\mathrm{abs}(X) := \begin{cases} Eva, & \text{if } X := \Lambda vab \quad \text{(an abstraction term)} \\ \text{undefined}, & \text{else.} \end{cases}$$

(That is: abs takes the abstraction prefix of any abstraction term into the corresponding assumption.)

14.9.COMMENT.

The intended interpretation of an E-sentence is "typing". I.e., if a,b are LA-terms then "Eab" reads "a has type b" or "b is a type of a", etc.

15. Contexts.

15.1. DEFINITION.

The sets Contx_n^{LA} ($n \in \mathbb{N}$) are the least sets such that

if $\bar{v}_n$ is in Svar_n and $\bar{a}_n$ is in Sterm_n^{LA} then $\underline{T}\bar{v}_n\bar{a}_n$ is in Contx_n^{LA}.

An element of Contx_n^{LA} is _a context of length_ n _in_ LA ($n \in \mathbb{N}$). (LA := PA, CA, etc.)

15.2. DEFINITION.

If $\Delta_n := \underline{T}\bar{v}_n\bar{a}_n$ is a context of length n in LA then _the variable-string_ of Δ_n is $\bar{v}_n$, while $\bar{a}_n$ is _its category-string_. Finally, the variable-string and the category- string of Δ_n are _the strings_ of Δ_n.

15.3. NOTATION.

We shall mainly use Δ_0 in order to denote _the context of length_ 0 _in_ any LA. (That is: $\Delta_0 := \underline{T}\underline{s}_0\square\underline{S}_0\square$ or $\Delta_0 := \underline{T}\boxtimes\boxtimes$.)

Whenever

$$\bar{v}_n := \langle v_1, \dots, v_n \rangle \quad \text{is in } \mathrm{Svar}_n$$

and

$$\bar{a}_n := \langle a_1, \dots, a_n \rangle \quad \text{is in } \mathrm{Sterm}_n^{LA},$$

we shall frequently write

$$\Delta_0[v_1 : a_1] \dots [v_n : a_n],$$

for $\underline{T}\bar{v}_n\bar{a}_n$, omitting sometimes Δ_0, if $n \geqslant 1$.

Further, for $n \geqslant 1$, Δ_n, possibly with superscripts will range on contexts of length n and the subscript "n" will be omitted whenever the length of the context is known.

15.4. NOTATION.

The following _epifunctions_ will be used in order to refer to the strings of a given context without mentioning them: $\underline{\underline{\underline{str}}}_1(\Delta_0) = \underline{s}_0\square$, $\underline{\underline{\underline{str}}}_2(\Delta_0) = \underline{S}_0\square$ and

$\underline{\underline{\underline{str}}}_1(\Delta_n) = \bar{v}_n$ if $\Delta_n := \underline{T}\bar{v}_n\bar{a}_n$ is in Contx_n^{LA}, else $\underline{\underline{\underline{str}}}_1$ is undefined,

$\underline{\underline{\underline{str}}}_2(\Delta_n) = \bar{a}_n$ if $\Delta_n := \underline{T}\bar{v}_n\bar{a}_n$ is in Contx_n^{LA}, else $\underline{\underline{\underline{str}}}_2$ is undefined ($n \geqslant 1$).

Sometimes, it will be also convenient to use _epifunctions_ $\underline{\underline{\underline{ass}}}_i^n$ ($n \in \mathbb{N}, 0 \leqslant i \leqslant n$) defined by

$$\underline{\underline{\underline{ass}}}_i^n(X) := \begin{cases} \Delta_0 & \text{if } X = \Delta_0, \\ \underline{E}v_i a_i & \text{if } X = \underline{T}\bar{v}\bar{a},\ \bar{v} := \langle v_1, \dots, v_n \rangle, \\ & \quad \bar{a} := \langle a_1, \dots, a_n \rangle, 1 \leqslant i \leqslant n \\ \text{undefined, else.} & \end{cases}$$

15.5.DEFINITION.

Let $\Delta_n := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ and $\Delta_m := \Delta_0[v_1':a_1']\ldots[v_m':a_m']$ be to contexts in LA ($n,m \in \mathbb{N}$).

Δ_n is <u>a subcontext of</u> Δ_m (<u>notation</u>: $\Delta_n \ \underline{\underline{sub}}\ \Delta_m$) if

(1) $n \leq m$,

(2) for all $i, 1 \leq i \leq n$, if $v_i \equiv \underline{\underline{elt}}_i^n(\underline{\underline{str}}_1(\Delta_n))$ then $v_i \equiv v_i'$,

(3) for all $i, 1 \leq i \leq n$, if $a_i \equiv \underline{\underline{elt}}_i^n(\underline{\underline{str}}_2(\Delta_n))$ then $a_i \equiv a_i'$

(Hence $\underline{\underline{str}}_j(\Delta_n)$ is a subsequence of $\underline{\underline{str}}_j(\Delta_m)$, $j = 1,2$; that is: the order of the elements in strings is preserved when passing from Δ_n to Δ_m).

15.6.REMARK.

Sometimes, contexts (of length $n, n \geq 1$) $\Delta_n := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ satisfying the following conditions below are called <u>telescopes</u> (<u>of length</u> n):

(1) $FV(a_1) = \emptyset$,

(2) $FV(a_{i+1}) \sqsubseteq \{v_1,\ldots,v_i\}$, for all $i, 1 \leq i < n$.

16. Constructions and sites.

16.1.DEFINITION.

(1) The sets $Pconstr_n^{LA}$ ($n \in \mathbb{N}$) are the least sets such that

if Δ_n is in $Contx_n^{LA}$ and Φ_n is in $Esent^{LA}$ with $\underline{sub}(\Phi_n)$ in $Pterm_n^{LA}$

then $\underline{D}\Delta_n\Phi_n\mathbf{D}$ is in $Pconstr_n^{LA}$,

provided $\underline{str}_1(\Delta_n) \equiv \underline{vtail}(\underline{sub}(\Phi_n))$;

(2) The sets $Dconstr_n^{LA}$ ($n \in \mathbb{N}$) are the least sets such that

if Δ_n is in $Contx_n^{LA}$ and Φ_n is in $Esent^{LA}$ with $\underline{sub}(\Phi_n)$ in $Dterm_n^{LA}$, and

" where $\underline{sub}(\Phi_n)$ ::= a "

is where-clause for LA, then $\underline{D}\Delta_n\Phi_n a$ is in $Dconstr_n^{LA}$,

provided $\underline{str}_1(\Delta_n) \equiv \underline{vtail}(\underline{sub}(\Phi_n))$

(3) An element of $Pconstr_n^{LA}$, $Dconstr_n^{LA}$ resp. is a (canonical) p-construction, resp. d-construction with length n in LA.

(4) $Constr_n^{LA} = Pconstr_n^{LA} \cup Dconstr_n^{LA}$ is the set of (canonical) constructions of length n in LA and the set

$$Constr^{LA} = \bigcup_{n \in \mathbb{N}} Constr_n^{LA}$$

is the set of (canonical) constructions in LA (or the set of LA-constructions).

16.2.NOTATION.

We also use the infinite unions $Pconstr^{LA}$, $Dconstr^{LA}$ defined from $Pconstr_n^{LA}$ and $Dconstr_n^{LA}$ as expected and an element of $Pconstr^{LA}$, $Dconstr^{LA}$ resp. will be called p-construction resp. d-construction in LA.

In general, p-constructions in LA will be denoted by

$$k_{\underline{p}} := \langle\!\langle \Delta_n ; \underline{p}(\overline{v}) : b \rangle\!\rangle,$$

where

$$\Delta_n := \underline{T}\overline{v}\overline{a}$$

$$\overline{v} := \underline{s}_n\mathbf{D}v_1 \ldots v_n$$

$$\overline{a} := \underline{S}_n\mathbf{D}a_1 \ldots a_n$$

and

$$\underline{p}(\overline{v}) := \delta_n\underline{p}v_1 \ldots v_n .$$

Similarly, a d-construction in LA will be denoted by a détour via the meta-language (epi-theory), viz., by restoring explicitely the corresponding "where-clauses". This gives:

$$k_{\underline{d}} \quad := \quad \langle\!\langle \Delta_n ; \underline{d}(\bar{v}) : b \rangle\!\rangle, \ \underline{\underline{where}} \ \underline{d}(\bar{v}) ::= a$$

with Δ_n, $\bar{v}$ and $\bar{a}$ as earlier and, moreover,

$$\underline{d}(\bar{v}) \ := \ \delta_{\underline{n}} dv_1 \ldots v_n, \qquad (n \in \mathbb{N}).$$

16.3. TERMINOLOGY.

Any LA-construction $k_{\underline{c}} := \underline{D}\Delta_n \Phi_n X$ (with either $X \equiv \square$ or X in $Term^{LA}$ satisfying the conditions of the definition above) has as immediate components

(1) its context-part Δ_n,

(2) its supporting (canonical) $\underline{E}$-sentence Φ_n and

(3) its definition-part X; the latter is the definiens of some definitional specification $\underline{\underline{df}}(k_{\underline{c}})$ say, for LA, such that $\underline{\underline{df}}(k_{\underline{c}})$ specifies $\underline{\underline{sub}}(\Phi_n)$.

Next, the remote components of $k_{\underline{c}}$ are defined by structural induction as follows:

(1r) if Δ_n is the context-part of $k_{\underline{c}}$ then

(11r) $\underline{\underline{str}}_1(\Delta_n)$ and $\underline{\underline{str}}_2(\Delta_n)$ are the strings of $k_{\underline{c}}$ and

(12r) for each i, $0 \leq i \leq n$,

$\underline{\underline{ass}}_i^n(\Delta_n)$ is an assumption of $k_{\underline{c}}$;

(2r) the strings and the assumptions of $k_{\underline{c}}$ are remote components of $k_{\underline{c}}$ (from Δ_n);

(22r) if $\bar{v}_n = \underline{\underline{str}}_1(\Delta_n)$ is a (variable-)string of $k_{\underline{c}}$ then each $\underline{\underline{elt}}_i^n(\bar{v}_n)$ $(0 \leq i \leq n)$ is a remote component of $k_{\underline{c}}$ (from Δ_n),

(23r) if $\bar{a}_n = \underline{\underline{str}}_2(\Delta_n)$ is a (term-)string of $k_{\underline{c}}$ then any quasi-sub-term of each $\underline{\underline{elt}}_i^n(\bar{a}_n)$, $0 \leq i \leq n$, is a remote component of $k_{\underline{c}}$ (from Δ_n);

(3r) if now Φ_n is the supporting (canonical) $\underline{E}$-sentence of $k_{\underline{c}}$ then

(31r) $\underline{\underline{sub}}(\Phi_n)$ is the subject or the definiendum of $k_{\underline{c}}$ and

(32r) $\underline{\underline{pred}}(\Phi_n)$ is the category part of $k_{\underline{c}}$;

(4r) the subject of $k_{\underline{c}}$ as well as any quasi-sub-term of its category-part are remote components of $k_{\underline{c}}$ (from Φ_n);

(41r) if $\underline{c}(\bar{v})$ is the subject of $k_{\underline{c}}$ then the head of $\underline{c}(\bar{v})$ is the identifier of $k_{\underline{c}}$ and the identifier of $k_{\underline{c}}$ is a remote component of $k_{\underline{c}}$ (from Φ_n);

(5r) if X is the definition-part of $k_{\underline{c}}$ and X is in $Term^{LA}$ then any quasi-sub-term of X, distinct from X itself, is a remote component of $k_{\underline{c}}$ (from its definition-part X).

The immediate and the remote components of a (canonical) LA-construction $k_{\underline{c}}$ are all and its only <u>components</u>.

16.4.REMARK.

If $k_{\underline{c}}$ is an LA-construction then no immediate component of $k_{\underline{c}}$ is a remote component of it and conversely.

For reasons that will be obvious later we shall introduce a <u>canonical parsing</u> of LA-constructions into <u>canonical components</u> (and <u>remote canonical components</u>) as follows:

16.5.TERMINOLOGY.NOTATION.

The <u>canonical components of</u> an LA-construction $k_{\underline{c}}$ are all and only the following components of $k_{\underline{c}}$:

(1) its <u>context-part</u>,hereafter denoted by $\underline{ctx}(k_{\underline{c}})$,

(2) its <u>category-part</u>,hereafter denoted by $\underline{cat}(k_{\underline{c}})$ and

(3) its <u>definition-part</u>,denoted by $\underline{def}(k_{\underline{c}})$.

The <u>remote canonical components of</u> $k_{\underline{c}}$ are all and only the following remote components of $k_{\underline{c}}$:

(1r) any remote component of $k_{\underline{c}}$ from $\underline{ctx}(k_{\underline{c}})$ is a remote canonical component of $k_{\underline{c}}$;

(2r) if $a \equiv \underline{def}(k_{\underline{c}})$ is in $Term^{LA}$ then any <u>proper</u> quasi-sub-term of a (i.e.,any quasi-sub-term of a,distinct from a itself) is a remote canonical component of $k_{\underline{c}}$,

(3r) any proper quasi-sub-term of $b \equiv \underline{cat}(k_{\underline{c}})$ is a remote canonical component of $k_{\underline{c}}$ and

(4r) the identifier of $k_{\underline{c}}$,hereafter denoted by $\underline{idf}(k_{\underline{c}})$ ($\equiv \underline{c}$),is a remote canonical component of $k_{\underline{c}}$.

16.6.REMARK.

It is easy to see that any component of an LA-construction $k_{\underline{c}}$,except its subject (definiendum),is either a canonical component of $k_{\underline{c}}$ or a remote canonical component of $k_{\underline{c}}$.

The latter kind of parsing will be useful in the tree-analysis of correctness in <u>3</u>. below.

16.7.REMARK.

We have introduced implicitely,in 16.5. above,several useful epifunctions (totally defined on $\mathrm{Constr}^{\mathrm{LA}}$).

For later purposes,it will be sufficient to have at hand only those named (explicitely) earlier,viz. ctx, cat, def and,possibly, idf.

Now we can introduce the sole text-category of LA,viz.

16.8.DEFINITION.

A site in LA (or an LA-site) is a finite set of (canonical) constructions in LA (for any LA,as above).

16.9.NOTATION.

We let $\mathrm{Site}^{\mathrm{LA}} := \mathcal{P}_{\omega}(\mathrm{Constr}^{\mathrm{LA}})$,where for any set $\underline{\underline{A}}$, $\mathcal{P}_{\omega}(\underline{\underline{A}})$ is the set of finite subsets of $\underline{\underline{A}}$.

$\mathrm{Site}^{\mathrm{LA}}$ is the category of sites in LA (or the category of LA-sites).

2. Language definition: correctness.

In this section we shall be concerned with the correctness part of the most important abstract AUT-languages (viz. PA,CA,QA and $Q^{-}A$).
The language definition method used in the sequel is very close to the pattern employed by D.van Daalen *et al*. in the so-called "E-definition" of the "reference"-AUT-languages AUT-68,AUT-QE (cf.,especially,VAN DAALEN 73-35,80-73).

20. Correctness categories.

We introduce the correctness categories of/in LA on an inductive set-theoretic pattern. (LA := PA,CA,QA,$Q^{-}A$,etc.).
In its most general form,a correctness category of/in LA is a relation on syntactic categories of/in LA.

20.1.DEFINITION.

For each LA (as above),the family of correctness categories of/in LA is a finite family of relations:

$$Corr^{LA} = \{Site_{\boxtimes}^{LA}, Contx_{\boxtimes}^{LA}, Esent_{\boxtimes}^{LA}, Term_{\boxtimes}^{LA}\}$$

such that the following (proper) inclusions hold:

(1) $Site_{\boxtimes}^{LA} \sqsubseteq Site^{LA}$,

(2) $Contx_{\boxtimes}^{LA} \sqsubseteq Contx^{LA} \times Site^{LA}$,

(3) $Esent_{\boxtimes}^{LA} \sqsubseteq Esent^{LA} \times Contx^{LA} \times Site^{LA}$,

(4) $Term_{\boxtimes}^{LA} \sqsubseteq Term^{LA} \times Contx^{LA} \times Site^{LA}$

and the elements of $Corr^{LA}$ are the least sets satisfying the correctness rules in Rl_{LA} below.

The correctness categories mentioned above are to be understood as being primitive for LA. In terms of these we can define derived correctness categories for LA and it will be useful to do so in order to shorten subsequent formulations.

20.2.DEFINITION.

For each LA, $Constr_{\boxtimes}^{LA}$ is the set of pairs $\langle k,B\rangle$ such that B is in $Site_{\boxtimes}^{LA}$ and k is in B.

The set-theoretic notation implicit in the above is not very convenient for the statement of the correctness rules in each Rl_{LA}.

Therefore, it will be appropriate to make use of some additional "syntactic sugar". The outcome will closely ressemble standard notational habits, as often used in the language-theory of "reference"-AUTOMATH (especially by D.van Daalen; cf. VAN DAALEN 72-28, 73-35, 80-73).

20.3. NOTATION.

$B,\Delta \vdash_{LA} a{:}b$ stands for $\langle \Phi,\Delta,B\rangle \in Esent_{\otimes}^{LA}$ where $\Phi := a{:}b$

$B,\Delta \Vdash_{LA} a$ stands for $\langle a,\Delta,B\rangle \in Term_{\otimes}^{LA}$

and

$B,\Delta \Vdash_{LA} a_1,\ldots,a_n$ stands for the conjunction of

$$B,\Delta \Vdash_{LA} a_i \quad (1 \leq i \leq n).$$

20.4. TERMINOLOGY.

It is somewhat awkward to provide a consistent and unambiguous reading in English for the elements of a correctness category of/in LA.

We shall, however, approximate this as follows:

(1) B _is_ _an_ LA-_compatible_ _site_ for "B is in $Site_{\otimes}^{LA}$",

(2) B _is_ LA-_compatible_ _and_ Δ _is_ LA-_admissible_ _for_ B for "$\langle\Delta,B\rangle$ is in $Contx_{\otimes}^{LA}$",

(3) B _is_ LA-_compatible_ _and_ k _is_ _sound_ _for/in_ B for "$\langle k,B\rangle$ is in $Constr_{\otimes}^{LA}$"

(4) Φ is _correct_ _for_ B _and_ Δ _such_ _that_ B _is_ LA-_compatible_ _and_ Δ _is_ LA-_admissible_ _for_ B for "$\langle\Phi,\Delta,B\rangle$ is in $Esent_{\otimes}^{LA}$",

(5) a _is_ _correct_ (or _t-correct_) _for_ B _and_ Δ _such_ _that_ B _is_ LA-_compatible_ _and_ Δ Δ _is_ LA-_admissible_ _for_ B for "$\langle a,\Delta,B\rangle$ is in $Term_{\otimes}^{LA}$".

Slightly elliptic variants of the above will be also adopted in colloquial ways of speaking/writing.

20.5.COMMENT.

We shall also introduce a couple of other "derived correctness categories in (for) LA" by a <u>détour</u> <u>via</u> the metatheory, using the associated reduction system $\underline{\underline{LA}}$ (with the appropriate "extensionality type" specified whenever necessary).

These "categories" will contain (by abuse of language) new kinds of "correct sentences" (or "correct formulas") in LA, besides the sets $CorrEsent^{LA} := Sec_1(Esent_{\boxtimes}^{LA})$. Cf. 05.4. for notation.

Specifically, one has

(1) <u>contraction-sentences</u> (or <u>C-sentences</u>) <u>in</u> LA,

(2) <u>reducibility-sentences</u> (or <u>R-sentences</u>) <u>in</u> LA and

(3) <u>convertibility-sentences</u> (or <u>Q-sentences</u>) <u>in</u> LA

and they are intended to "formalize" resp. the appropriate restrictions of $\underline{\underline{contr}}_{LA}$, $\underline{\underline{red}}_{LA}$ and $\underline{\underline{conv}}_{LA}$ to the sets $CorrTerm^{LA} := Sec_1(Term_{\boxtimes}^{LA})$.

Accurately, they must be thought of as being <u>relations</u>

$$\underline{X}sent_{\boxtimes}^{LA} \sqsubseteq (Term^{LA} \times Term^{LA}) \times Contx^{LA} \times Site^{LA}$$

where $X := \underline{C}, \underline{R}, \underline{Q}$ such that

$$\langle\langle a,b\rangle,\Delta,B\rangle \in \underline{X}sent_{\boxtimes}^{LA}$$

iff

(i) B is in $Site_{\boxtimes}^{LA}$,

(ii) $\langle\Delta,B\rangle$ is in $Contx_{\boxtimes}^{LA}$ and

(iii) $\langle a,\Delta,B\rangle$ and $\langle b,\Delta,B\rangle$ are both in $Term_{\boxtimes}^{LA}$

and such that

(iv) $a \;\underline{\underline{R}}\; b$ holds,

where

(1') $\underline{\underline{R}} := \underline{\underline{contr}}_{LA}$ if $X := \underline{C}$,

(2') $\underline{\underline{R}} := \underline{\underline{red}}_{LA}$ if $X := \underline{R}$ and

(3') $\underline{\underline{R}} := \underline{\underline{conv}}_{LA}$ if $X := \underline{Q}$.

In some presentations of the "reference"-AUT-languages at least something analogous to $\underline{Q}sent_{\boxtimes}^{LA}$ above was taken as a <u>primitive</u> <u>correctness</u> <u>category</u> (this approach was also mentioned in 05.2. and 10.1.; see e.g., VAN DAALEN 73-35, 80-72). In JUTTING 82-83 something similar to $\underline{R}sent_{\boxtimes}^{LA}$ and $\underline{C}sent_{\boxtimes}^{LA}$ was used in the language definition of several "reference"-AUT-SYNT languages.

Alternatively, one might think of the corresponding new ingredients as being mere abbreviational devices intended to simplify more involved ways of speking (and notation).

Here we shall adopt the latter point of view.

20.6.DEFINITION.

$Csent^{LA}_{\boxtimes}$ is the least set such that,for a,b in $Term^{LA}$,Δ in $Contx^{LA}$ and B in the category $Site^{LA}$,

$$\langle\langle a,b\rangle,\Delta,B\rangle \text{ is in } Csent^{LA}_{\boxtimes}$$

if

(1) B is in $Site^{LA}_{\boxtimes}$ and $\langle\Delta,B\rangle$ is in $Contx^{LA}_{\boxtimes}$,

(2) $B,\Delta \Vdash_{LA} a,b$

(3) $a\ \underline{\underline{contr}}_{LA}\ b$

20.7.NOTATION.

If B is in $Site^{LA}$ and Δ is in $Contx^{LA}$ then

$B,\Delta \ \vDash_{LA}\ a\ \underline{\underline{contr}}\ b$ or $B,\Delta \ \vDash\ a\ \underline{\underline{contr}}_{LA}\ b$

stands for

" $\langle\langle a,b\rangle,\Delta,B\rangle$ is in $Csent^{LA}_{\boxtimes}$ "

and it is understood that B is LA-compatible and Δ is LA-admissible for B (in current notation,the latter kind of information will be oft made explicit). Also we write,elliptically,

$$a\ \underline{\underline{contr}}^{\boxtimes}_{LA}\ b$$

if,for <u>some</u> LA-compatible B and <u>some</u> LA-admissible context Δ for B,one has

$B,\Delta \ \vDash_{LA}\ a\ \underline{\underline{contr}}\ b$.

That is: $\underline{\underline{contr}}^{\boxtimes}_{LA}$ is the restriction of $\underline{\underline{contr}}_{LA}$ to the corresponding set $CorrTerm^{LA} \times CorrTerm^{LA}$ (where $CorrTerm^{LA} := Sec_1(Term^{LA}_{\boxtimes})$;cf. 05.4.).

In the same conditions as above we write

$B,\Delta \ \vDash_{LA}\ a\ \underline{\underline{conv}}^1\ b$ (or even $B,\Delta \ \vDash\ a\ \underline{\underline{conv}}^1_{LA}\ b$)

for

"$B,\Delta \ \vDash_{LA}\ a\ \underline{\underline{contr}}\ b$ <u>or</u> $B,\Delta \ \vDash_{LA}\ b\ \underline{\underline{contr}}\ a$",

and adopt the same elliptical notation

$$a\ \underline{\underline{conv}}^1_{LA}\ b$$

for cases where

$B,\Delta \ \vDash_{LA}\ a\ \underline{\underline{conv}}^1\ b$

holds for <u>some</u> LA-compatible site B and <u>some</u> context Δ ,LA-admissible for B.

20.8.DEFINITION.

$Rsent^{LA}_{\boxtimes}$ is the least set such that,for all a,b in $Term^{LA}$,all Δ in $Contx^{LA}$ and all B in $Site^{LA}$,

$$\langle\langle a,b\rangle,\Delta,B\rangle \text{ is in } Rsent^{LA}_{\boxtimes}$$

if

(1) B is in $\mathrm{Site}^{LA}_{\boxtimes}$ and $\langle\Delta,B\rangle$ is in $\mathrm{Contx}^{LA}_{\boxtimes}$ and

(2) there are LA-terms

$$a \equiv a_0, a_1, \ldots, a_n, a_{n+1} \equiv b$$

such that

$$B,\Delta \models a_i \underline{\underline{\mathrm{contr}}}_{LA}\, a_{i+1} \qquad (0 \leq i \leq n).$$

20.9.DEFINITION.

$\mathrm{Qsent}^{LA}_{\boxtimes}$ is the least set such that,for all a,b in Term^{LA},all Δ in Contx^{LA} and all B in Site^{LA},

$$\langle\langle a,b\rangle,\Delta,B\rangle \text{ is in } \mathrm{Qsent}^{LA}_{\boxtimes}$$

if

(1) B is in $\mathrm{Site}^{LA}_{\boxtimes}$ and $\langle\Delta,B\rangle$ is in $\mathrm{Contx}^{LA}_{\boxtimes}$ and

(2) there are LA-terms

$$a \equiv a_0, a_1, \ldots, a_n, a_{n+1} \equiv b$$

such that

$$B,\Delta \models a_i \underline{\underline{\mathrm{conv}}}^{1}_{LA}\, a_{i+1} \qquad (0 \leq i \leq n).$$

20.10.REMARK.

By the conventions adopted in 20.7. above,the LA-terms a_i $(0 \leq i \leq n+1)$ in 20.8. and 20.9. have to be such that

$$B,\Delta \Vdash a_i .$$

20.11.NOTATION.

We shall introduce analogues of the abbreviations in 20.7. as follows: let B be an LA-site and Δ be an LA-context,then

(1) $\qquad B,\Delta \models a \underline{\underline{\mathrm{red}}}_{LA}\, b$

resp.

(2) $\qquad B,\Delta \models a \underline{\underline{\mathrm{conv}}}_{LA}\, b$

stand for

(1') $\qquad$ " $\langle\langle a,b\rangle,\Delta,B\rangle$ is in $\mathrm{Rsent}^{LA}_{\boxtimes}$ "

and

(2') $\qquad$ " $\langle\langle a,b\rangle,\Delta,B\rangle$ is in $\mathrm{Qsent}^{LA}_{\boxtimes}$ "

resp.It follows that B is LA-compatible and Δ is LA-admissible for B,but the latter kind of information will be often made (somewhat redundantly) explicit. If,for <u>some</u> LA-compatible site B and <u>some</u> LA-context Δ ,LA-admissible for B, we have (1) or (2) above we also write elliptically

(1") $\qquad a \underline{\underline{\mathrm{red}}}^{\boxtimes}_{LA}\, b$

or

(2") $a\ \underline{\underline{conv}}^{\boxtimes}_{LA}\ b$,

resp.

Clearly, $\underline{\underline{red}}^{\boxtimes}_{LA}$ and $\underline{\underline{conv}}^{\boxtimes}_{LA}$ resp. are the restrictions of $\underline{\underline{red}}_{LA}$,$\underline{\underline{conv}}_{LA}$ resp. to the appropriate sets

$$CorrTerm^{LA} \times CorrTerm^{LA}$$

where $CorrTerm^{LA}$ is as earlier (in 20.7.).

20.12.COMMENT.

As used here, $\underline{\underline{conv}}^{\boxtimes}_{LA}$ corresponds to D.van Daalen's "$\underline{Q}$" in VAN DAALEN 80-73, except that, in the latter case, "$\underline{Q}$" is a primitive constructor in the language, while, here, we have preferred to keep the matters concerning reduction and definitional equality at a meta-linguistic level.

Before going into the details of the correctness rules in each set Rl_{LA}, we shall list some more notational conventions and terminology to be used later on.

20.13.NOTATION.

Let B be an LA-compatible site and Δ be an LA-admissible context for B. Then, for all LA-terms $a_0, a_1, \ldots, a_n, a_{n+1}$, the list of epi-theoretic statements

$$B,\Delta \vdash_{LA} a_i : a_{i+1} \qquad (0 \leq i \leq n)$$

will be condensed, for convenience, in a single line, as follows:

$$B,\Delta \vdash_{LA} a_0 : a_1 : \ldots : a_n : a_{n+1}.$$

20.14.TERMINOLOGY.NOTATION.

Let B be an LA-site and Δ be an LA-context with $\underline{\underline{lh}}(\Delta) = n, n \geq 0$.

(1) A variable v is said to be <u>fresh for</u> Δ if it does not occur in Δ as a sub-word of it.

(2) Define

$$\underline{\underline{idf}}(B) = \{\underline{\underline{idf}}(k) : k \text{ in } B\}.$$

A floating constant $\underline{c}$ is <u>fresh for</u> B if it is not in $\underline{\underline{idf}}(B)$.

(3) Let $\bar{a}_n := \langle a_1, \ldots, a_n \rangle$ be in $Sterm^{LA}_n$ $(n \geq 0)$.

Then a floating constant $\underline{c}$ is <u>rank-fresh for</u> $\bar{a}_n$ if it is rank-fresh for each a_i in $\bar{a}_n$.

(4) In analogy with (3) one may say $\underline{c}$ is <u>minimally fresh for</u> $\bar{a}_n$ if

(41) $\underline{\underline{rank}}(\underline{c}) = 0$, whenever $n = 0$ and

(42) $\underline{\underline{rank}}(\underline{c}) = \underline{\underline{max}}\ \{\underline{\underline{rank}}^{+}(a_i) : 1 \leq i \leq n\} + 1$, whenever $n \geq 1$.

21. Correctness for Primitive and Classical (Abstract) AUTOMATH.

In this section the definition of PA (= Abstract Primitive AUTOMATH) and CA (= Abstract Classical AUTOMATH) will be completed by specifying the appropriate sets of correctness rules (Rl_{PA} and Rl_{CA} resp.).

Actually, PA is a sub-language of CA (or, if one prefers, CA is an extension of PA) and the corresponding correctness rules will be chosen such as to reflect straightforwardly this situation (i.e., we will indeed have Rl_{PA} strictly contained in Rl_{CA}, qua sets).

21.1. COMMENT.

In detail, the "structure" of the sets Rl_{LA} (LA := PA, CA) is as follows:

(1) structural rules concerning

(11) the compatibility of sites in LA: (Si), (Sr-1p), (Sr-2p), (Sr-1d), (Sr-2d),

(12) the admissibility of contexts in LA: (Ci), (Cr-1), (Cr-2);

(2) basic rules concerning

(21) the correctness of E-sentences in LA: (Ei), (Er-c),

(22) the correctness of LA-terms in LA: (Ti), (Tr);

(3) rules of category conversion (for LA), viz. rules of

(31) atomic reduction for categories in LA: (CC_1) and

(32) atomic expansion for categories in LA: (CC_2).

These rules will be also present in the languages $Q^{-}A$ and QA (to be studied in 22. below).

Besides the above, Rl_{CA} contains also:

(4) specific CA-rules, concerning the correctness of E-sentences on CA (involving applications and abstractions), viz.

(41) an application rule: (app-1) and

(42) abstraction rules: (abs-1-CA), (abs-2).

Accurately, in the case of Rl_{CA}, we are defining two distinct abstract AUT-languages by the same set of correctness rules, according to the "extensionality type" of the corresponding relation of "definitional equality" taken as primitive in the language (or in the associated reduction system): viz. we have β-CA, with $contr_{\beta\delta}$, $red_{\beta\delta}$ and $conv_{\beta\delta}$ in CA, and the "extensional version" of the language, namely $\beta\eta$-CA, with $contr_{\beta\eta\delta}$, $red_{\beta\eta\delta}$ and $conv_{\beta\eta\delta}$ in the associated reduction system.

211. Correctness rules for PA and CA.

We state first the structural rules in Rl_{PA}, Rl_{CA} (and Rl_{QA}, in fact). The notation and terminology used here is as earlier. The assumption of freshness for variables will be always made explicit (this is somewhat redundant, however).

I. STRUCTURAL RULES

$LA := PA, CA, (Q^{-}A, QA)$

I.1. $Site^{LA}_{\boxtimes}$-RULES

I.1.1. $Site^{LA}_{\boxtimes}$-INITIALIZATION.

(Si) The empty site is LA-compatible.

I.1.2. $Site^{LA}_{\boxtimes}$-RECURSION.

Let $\Delta_n := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ $(n \geqslant 0)$

$\underline{p}(\overline{v}) := \underline{p}(v_1,\ldots,v_n)$, $\underline{p}$ in Pfl^{LA}_n

$\underline{d}(\overline{v}) := \underline{d}(v_1,\ldots,v_n)$, $\underline{d}$ in Dfl^{LA}_n

and, where

$\underline{c}(\overline{v}) := \underline{p}(\overline{v})$ or $\underline{c}(\overline{v}) := \underline{d}(\overline{v})$, with $\underline{c}$ in Fl^{LA}_n,

and a in $Term^{LA}$,

$k^a_{\underline{c}} := \langle \Delta_n; \underline{c}(\overline{v}):a \rangle$ (with $k^a_{\underline{c}}$ in $Constr^{LA}_n$, $n \geqslant 0$)

Assume B is an LA-compatible site and Δ_n is an LA-admissible context for B.

(Sr-1p) If $\underline{p}$ is fresh for B and rank-fresh for $\overline{a} := \langle a_1,\ldots,a_n \rangle$

then

$B \sqcup \{k^{\tau}_{\underline{p}}\}$ is LA-compatible,

with $k^{\tau}_{\underline{p}} := \langle \Delta_n; \underline{p}(\overline{v}):\tau \rangle$.

(Sr-2p) If $\underline{p}$ is fresh for B,

$B,\Delta_n \vdash_{LA} a:\tau$ and

$\underline{p}$ is rank-fresh for $\bar{a}' := \langle a_1,\dots,a_n,a \rangle$

then

$B \sqcup \{k^a_{\underline{p}}\}$ is LA-compatible ,

with $k^a_{\underline{p}} := \langle\!\langle \Delta_n;\ \underline{p}(\bar{v}):a \rangle\!\rangle$.

(Sr-1d) If $\underline{d}$ is fresh for B,

$B,\Delta_n \vdash_{LA} a:\tau$ and

$\underline{d}$ is rank -fresh for $\bar{a}' := \langle a_1,\dots,a_n,a \rangle$

then

$B \sqcup \{k^\tau_{\underline{d}}\}$ is LA-compatible,

with $k^\tau_{\underline{d}} := \langle\!\langle \Delta_n;\ \underline{d}(\bar{v}):\tau \rangle\!\rangle$, where $\underline{d}(\bar{v}) ::= a$.

(Sr-2d) If $\underline{d}$ is fresh for B,

$B,\Delta_n \vdash_{LA} b:a:\tau$ and

$\underline{d}$ is rank-fresh for $\bar{a}'' := \langle a_1,\dots,a_n,a,b \rangle$

then

$B \sqcup \{k^a_{\underline{d}}\}$ is LA-compatible,

with $k^a_{\underline{d}} := \langle\!\langle \Delta_n;\ \underline{d}(\bar{v}):a \rangle\!\rangle$,where $\underline{d}(\bar{v}) ::= b$.

I.2.$\text{Contx}^{LA}_{\boxtimes}$-RULES.

I.2.1.$\text{Contx}^{LA}_{\boxtimes}$-INITIALIZATION.

(Ci) If B is LA-compatible

then

Δ_0 is LA-admissible for B.

I.2.2.$Contx_{\boxtimes}^{LA}$-RECURSION.

Let $\Delta_n := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ and v_{n+1} be in Var.

Assume B is an LA-compatible site and Δ_n is an LA-admissible context for B.

(Cr-1) If v_{n+1} is fresh for Δ_n
then
$\Delta^{\tau}_{n+1} := \Delta_n[v_{n+1}:\tau]$ is LA-admissible for B.

(Cr-2) If v_{n+1} is fresh for Δ_n,
$B,\Delta_n \vdash_{LA} a:\tau$ and
(v_{n+1} is not in FV(a))
then
$\Delta^{a}_{n+1} := \Delta_n[v_{n+1}:a]$ is LA-admissible for B.

Next we state the <u>basic</u> <u>correctness</u> <u>rules</u> for PA and CA (resp. $Q^{-}A$,QA;cf. below).
They concern the <u>correctness</u> <u>of</u> <u>E</u>-sentences ϕ in LA (:= PA,CA,etc.) where
<u>sub</u>(ϕ) is either <u>a</u> <u>variable</u> or a <u>head-term</u> and the <u>correctness</u> <u>of</u> LA-<u>terms</u>
(which,in fact,is a concept derived from the former one).
As variables and head-terms are supposed to be present in any LA,these rules
(or something similar) are,again,common to all abstract AUT-languages.

II.BASIC RULES.

LA := PA,CA ($Q^{-}A$,QA,etc.)

II.1.$Esent_{\boxtimes}^{LA}$-RULES.

II.1.1.$Esent_{\boxtimes}^{LA}$-INITIALIZATION.

Let $\Delta_n := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ $(n \geqslant 1)$.

(Ei) If B is LA-compatible and
Δ_n is LA-admissible for B
then
$B,\Delta_n \vdash_{LA} v_i:a_i$ for <u>all</u> $i, 1 \leqslant i \leqslant n$.

II.1.2.$\mathrm{Esent}_{\boxtimes}^{LA}$-RECURSION.

Let $\Delta_n \;:=\; \Delta_0[v_1:a_1]\ldots[v_n:a_n] \qquad (n \geq 0),$

$\Delta'_m \;:=\; \Delta_0[v'_1:a'_1]\ldots[v'_m:a'_m] \qquad (m \geq 0),$

and, for $\underline{c}$ in Fl_n^{LA}, (and fresh for Δ_n and a in Term^{LA})

$\underline{c}(\overline{v}) \;:=\; \underline{c}(v_1,\ldots,v_n)$

$k_{\underline{c}}^{a} \;:=\; \langle\!\langle \Delta_n;\ \underline{c}(\overline{v}):a \rangle\!\rangle,$

and, for $\overline{b} \;:=\; \langle b_1,\ldots,b_n \rangle,$

$\underline{c}(\overline{b}) \;:=\; (\underline{c}(\overline{v}))[\![\overline{b} := \overline{v}]\!].$

(Er-c) If B is LA-compatible,

Δ_n is LA-admissible for B,

$k_{\underline{c}}^{a}$ is (sound) in B and

$B,\Delta'_m \vdash_{LA} b_i : a_i[\![\overline{b} := \overline{v}]\!] \qquad$ (<u>all</u> $i, 1 \leq i \leq n$),

such that

Δ'_m is LA-admissible for B,

then

$B,\Delta'_m \vdash_{LA} \underline{c}(\overline{b}) : a[\![\overline{b} := \overline{v}]\!].$

II.2.$\mathrm{Term}_{\boxtimes}^{LA}$-RULES.

II.2.1.$\mathrm{Term}_{\boxtimes}^{LA}$-INITIALIZATION.

(Ti) If B is LA-compatible and

Δ_n is LA-admissible for B

then

$B,\Delta_n \Vdash_{LA} \tau.$

II.2.2.$\mathrm{Term}_{\boxtimes}^{LA}$-RECURSION.

(Tr) If B is LA-compatible,

Δ_n is LA-admissible for B and

$B,\Delta_n \vdash_{LA} b:a$

then

$B,\Delta_n \Vdash_{LA} b.$

III.RULES OF CATEGORY CONVERSION.

LA := PA,CA,($Q^{-}A$,QA,etc.)

Assume B is an LA-compatible site and Δ_n is an LA-admissible context for B.

III.1. ATOMIC REDUCTION FOR CATEGORIES.

(CC_1) If $B,\Delta_n \vdash_{LA} a:b_1$,

$B,\Delta_n \Vdash_{LA} b_1,b_2$ and

b_1 $\underline{\underline{contr}}_{LA}$ b_2

then

$B,\Delta_n \vdash_{LA} a:b_2$.

III.2. ATOMIC EXPANSION FOR CATEGORIES.

(CC_2) If $B,\Delta_n \vdash_{LA} a:b_1$,

$B,\Delta_n \Vdash_{LA} b_1,b_2$ and

b_2 $\underline{\underline{contr}}_{LA}$ b_1

then

$B,\Delta_n \vdash_{LA} a:b_2$.

NOTE.Here $\underline{\underline{contr}}_{LA}$:= $\underline{\underline{contr}}_{\delta}$,if LA := PA,and either $\underline{\underline{contr}}_{LA}$:= $\underline{\underline{contr}}_{\beta\delta}$ or $\underline{\underline{contr}}_{LA}$:= $\underline{\underline{contr}}_{\beta\eta\delta}$,else.So if LA is not PA,one defines β-LA with $\underline{\underline{contr}}_{\beta\delta}$ and $\beta\eta$-LA with $\underline{\underline{contr}}_{\beta\eta\delta}$.

211.1.COMMENT.

This completes the definition of PA.So,for further reference,Rl_{PA} contains all and only the following rules (as primitive correctness rules for PA):

I. (Si),(Sr-1p),(Sr-2p),(Sr-1d),(Sr-2d);(Ci),(Cr-1),(Cr-2);

II. (Ei),(Er-c);(Ti),(Tr);

III. (CC_1),(CC_2).

Clearly,with the exception of (CC_1) and (CC_2),the remaining rules were classified according to the correctness categories of their outputs (that is: the $Site^{LA}_{\boxtimes}$-rules "produce" <u>new</u> LA-compatible sites,similarly,the $Contx^{LA}_{\boxtimes}$-rules produce <u>new</u> LA-admissible contexts and so on).

As intended,20.1. (just completed above,for the case LA := PA) is a usual <u>inductive definition</u>.In particular,even if the underlying recursion relies on a <u>simultaneous induction</u>,each of the recursive sets thereby defined is invariably given by

(1) some initial clause ("initialization"-clause: (Si),(Ci),(Ei),(Ti))

and

(2) several "recursion"-clauses (four for sites,two for contexts,one for E-sentences and one for terms).

Only the $Site^{LA}_{\alpha}$-rules are "building" rules;the remaining ones serve to "transfer" information from old LA-compatible sites to new ones (one would even want to call them "transport"-rules).

In the Rules of Category Conversion the appellation "category" is to be understood as referring to the "category-part" of an LA-construction (and <u>not</u> to "syntactic" or "correctness categories" in LA).

The latter do not only "transfer" some information from some LA-site to another one,but also "modify" the information concerned by a <u>détour via</u> the reduction system (here PA) associated to the language.

<u>Without</u> (CC_i),i = 1,2, the language definition is,in fact,<u>purely syntactic</u> (or,if one prefers,<u>combinatorial in nature</u> - in the sense of "combinatorics"; see also 3 below),but also <u>without</u> these rules there is hardly some interest in the corresponding sub-lànguage(s).They clearly introduce the <u>sui generis combinatory aspect</u> of the language under consideration (where "combinatory" is to be taken somewhat <u>à la</u> KLOP 80,as qualifying a <u>specific reduction system</u>,while,if LA is not PA,its connotation is very close to Curry's "combinatory logic").

The import of the restrictions of "rank - freshness" on floating constants in the $Site^{LA}_{\alpha}$-recursion rules will appear later on.

211.2.TERMINOLOGY.REMARK.

In any $Site^{LA}_{\alpha}$-recursion rule (viz. (Sr-1p),(Sr-2p),(Sr-1d) and (Sr-2d) above), the LA-construction by which some LA-compatible site B is "extended" will be said to be <u>correct for</u> B <u>in</u> LA or even B-<u>correct in</u> LA.

Now,obviously,if some k is B-correct in LA,for some LA-compatible site B, then $B \sqcup \{k\}$ is LA-compatible and k is LA-sound in $B \sqcup \{k\}$.Conversely,if some k is sound in some LA-compatible site B,then <u>it is not true</u> that k is B-correct in LA (since idf(k) is not fresh for B any more).

(Compare with the "correctness of lines w.r.t. correct books" in the "reference" versions of the AUT languages.)

Now the definition of CA(-correctness) will be completed by listing the remaining rules in Rl_{CA} (which are not in Rl_{PA}). According to the classification mentioned in 211.1. above the rules following below are $Esent_{\boxtimes}^{CA}$-rules, i.e., they will have outputs in the correctness category $Esent_{\boxtimes}^{CA}$.

IV. SPECIFIC RULES FOR CA.

IV.1. APPLICATION RULE 1.

LA := CA ($Q^{-}A$, QA, etc.)

(app-1) If B is LA-compatible,

Δ_n is LA-admissible for B,

$B,\Delta_n \vdash_{LA} a':a:\tau$ and

$B,\Delta_n \vdash_{LA} b:[v:a]b'$

then

$B,\Delta_n \vdash_{LA} \{a'\}b:b'[\![a' := v]\!]$.

IV.2. ABSTRACTION RULES.

IV.2.1. ABSTRACTION RULE 1 FOR CA.

Let $\Delta_n := \Delta_0[v_1:a_1]\dots[v_n:a_n]$

and v_{n+1} be in Var.

(abs-1-CA) If B is CA-compatible,

Δ_n is CA-admissible,

$B,\Delta_n \vdash_{CA} a:\tau$

$\Delta_{n+1}^{a} := \Delta_n[v_{n+1}:a]$ is CA-admissible for B and

$B,\Delta_{n+1}^{a} \vdash_{CA} b:\tau$

then

$B,\Delta_n \vdash_{CA} [v_{n+1}:a]b : \tau$.

IV.2.2.ABSTRACTION RULE 2.

$LA := CA$ ($\bar{Q}A$, QA, etc.).

Let $\Delta_n := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$

and v_{n+1} be in Var.

(abs-2) If B is LA-compatible,

Δ_n is LA-admissible for B,

$B,\Delta_n \vdash_{LA} a:\tau$,

$\Delta_{n+1} := \Delta_n[v_{n+1}:a]$ is LA-admissible for B,

$B,\Delta_{n+1} \vdash_{LA} b':b:\tau$

then

$$B,\Delta_n \vdash_{LA} [v_{n+1}:a]b' : [v_{n+1}:a]b.$$

211.3.COMMENT.

For further reference, Rl_{CA}, the set of primitive correctness rules of CA contains the rules in Rl_{PA} together with the following three $\mathrm{Esent}_{\boxtimes}^{CA}$-rules: IV.(app-1), (abs-1-CA), (abs-2).

In particular, rule (abs-1-CA) is <u>prima facie</u> a peculiar feature of CA, when comparing the latter with other extensions of PA. Indeed, $\bar{Q}A$, discussed incidentally below, lacks (abs-1-CA) as a <u>derived rule</u>, but it is quickly regained in QA by the mechanism of <u>category inclusion</u>. So what is rather <u>specific</u> to CA - when compared with bigger games - is that (abs-1-CA) <u>forbids</u> actually (CA-) correct terms of the form $[v_1:a_1][v_2:a_2]\ldots[v_n:a_n]\tau$ $(n \geqslant 1)$.

212. Basic epi-theory for PA, CA: "structural" lemmas.

We examine now some basic properties of the correctness categories in PA and CA. Let us first give names to some sets used informally earlier.

212.1.NOTATION.COMMENT.

Where Sec_1 is as introduced in 05.4. above and LA := PA,CA,we set:

(1) $CorrContx^{LA} = Sec_1(Contx^{LA}_{\boxtimes})$.

(2) $CorrConstr^{LA} = Sec_1(Constr^{LA}_{\boxtimes})$.

(3) $CorrEsent^{LA} = Sec_1(Esent^{LA}_{\boxtimes})$.

(4) $CorrTerm^{LA} = Sec_1(Term^{LA}_{\boxtimes})$.

That is,informally: $CorrContx^{LA}$ is the set of LA-admissible contexts for LA-sites B such that B is LA-compatible (the set of "correct contexts in LA"), $CorrConstr^{LA}_{\boxtimes}$ is the set of LA-sound constructions in LA-sites B such that B is LA-compatible (the set of "correct constructions in LA"),$CorrEsent^{LA}$ is the set of "correct <u>E</u>-sentences in LA" and $CorrTerm^{LA}$ is the set of "correct LA-terms". E.g.,in other words,

$CorrTerm^{LA}$ is the least subset of LA-terms a such that

$$B,\Delta \Vdash_{LA} a,$$

for some LA-compatible B and some LA-admissible context Δ for B.

It is appropriate to start with the remark that (for LA := PA,CA) $CorrEsent^{LA}$ determines a partition of $CorrTerm^{LA}$ in (<u>three</u>) pairwise disjoint equivalence classes.Using a standard terminology in "reference" AUTOMATH we call these equivalence classes "degrees" and introduce them by recursion as follows. Hereafter,if not otherwise specified,LA := PA or CA.

212.2.DEFINITION.

Let B be an LA-compatible site and Δ be an LA-admissible context for B. For any LA-term a,if

$$B,\Delta \Vdash_{LA} a,$$

then <u>the degree of</u> a (<u>relative to</u> B <u>and</u> Δ <u>in</u> LA),denoted by $\underline{\underline{dg}}_{LA}(B,\Delta,a)$, is defined by:

(1) $\underline{\underline{dg}}_{LA}(B,\Delta,a) = 1$ if $a \equiv \tau$.

(2) If $\underline{\underline{dg}}_{LA}(B,\Delta,b) = n$ and $B,\Delta \vdash_{LA} a:b$ then $\underline{\underline{dg}}_{LA}(B,\Delta,a) = n+1$.

The subscript "LA" will be always omitted from notations indicating degrees. The following fact motivates some further simplifications in the (meta-)notation for degrees.

212.3. REMARK.

For LA-sites B,B',LA-contexts Δ,Δ' and LA-terms a not in Var such that:

(1) B,B' are LA-compatible,

(2) Δ is LA-admissible for B,

(3) Δ' is LA-admissible for B',

(4) $B,\Delta \Vdash_{LA} a$ and

(5) $B',\Delta' \Vdash_{LA} a$

one has also

(6) $\underline{\underline{dg}}(B,\Delta,a) = \underline{\underline{dg}}(B',\Delta',a)$.

It is easy to see that the above does not obtain for a in Var.

However, the following notation can be used rather freely.

212.4. NOTATION.

For all LA-terms a, we write "$\underline{\underline{dg}}(a) = n$" for "$\underline{\underline{dg}}(B,\Delta,a) = n$", leaving (the LA-site) B and (the LA-context) Δ unspecified, if this does not lead to confusions.

212.5. LEMMA.

(LA := PA,CA,etc. See 22. below.)

For all B,Δ such that B is LA-compatible and Δ is LA-admissible for B, and all LA-terms a such that $B,\Delta \Vdash_{LA} a$, one has

$$1 \leqslant \underline{\underline{dg}}(B,\Delta,a) \leqslant 3.$$

Proof. By induction on LA-correctness.⊠

212.6. REMARK.

The following facts can be checked easily: let B be an LA-compatible site and Δ be an LA-admissible context for B.

(1) For all LA-terms a,b, if $B,\Delta \vdash_{LA} a{:}b$ then $\underline{\underline{dg}}(a) \neq 1$. Indeed, in this case one has

$$\underline{\underline{dg}}(B,\Delta,a) = \underline{\underline{dg}}(B,\Delta,b) + 1,$$

and there is no degree 0 (zero) in Definition 212.2.

(2) For all LA-terms a,b, if $B,\Delta \models a\ \underline{\underline{R}}\ b$, where $\underline{\underline{R}} := \underline{\underline{contr}}_{LA}, \underline{\underline{red}}_{LA}, \underline{\underline{conv}}_{LA}$ then

$$\underline{\underline{dg}}(B,\Delta,a) = \underline{\underline{dg}}(B,\Delta,b).$$

(3) In particular, if a is a head-term in LA, with $\underline{\underline{lh}}(a) = n, n \geqslant 1$, then for all $i, 1 \leqslant i \leqslant n$,

$$2 \leqslant \underline{\underline{dg}}(\underline{\underline{arm}}_i^n(a)) \leqslant 3.$$

(4) By clause (1) of 212.2. the only LA-term a with $\underline{\underline{dg}}(a) = 1$ is τ so there are no variables, no head-terms, no application-terms and no abstraction-terms in LA (:= PA,CA) with degree 1.

We are now going to establish some "structural lemmas" on term-correctness in PA,CA which will be oft employed later.

212.7.LEMMA.

If B is LA-compatible, Δ is LA-admissible for B and, for v in Var,

$$B,\Delta \Vdash_{LA} v$$

then <u>either</u>

(1) $$B,\Delta \vdash_{LA} v{:}\tau$$

<u>or</u>

(2) for some LA-term a, one has

$$B,\Delta \vdash_{LA} v{:}a{:}\tau.$$

<u>Proof</u>. By induction on LA-correctness.⊠

212.8.REMARK.

In other words, the preceeding Lemma amounts to the fact that, for any variable v in CorrTerm^{LA}, one has $2 \leqslant \underline{\underline{dg}}(v) \leqslant 3$ (such v's are "free").

The next lemma is somewhat more informative.

212.9.LEMMA.

Let B be LA-compatible and Δ be LA-admissible for B. If

$$B,\Delta \Vdash_{LA} v, \qquad \text{for v in Var,}$$

then

(1) for some $n, n \geqslant 1, \underline{\underline{lh}}(\Delta) = n$ and

(2) for some $i, 1 \leqslant i \leqslant n$,

(21) $$v \equiv \underline{\underline{sub}}(\underline{\underline{ass}}_i^n(\Delta))$$

and, moreover, (for the same $i, 1 \leqslant i \leqslant n$),

(22) $$B,\Delta \vdash_{LA} v{:}a_i$$

with $a_i \equiv \underline{\underline{pred}}(\underline{\underline{ass}}_i^n(\Delta))$.

Proof. In each case (LA := PA,CA), by induction on the definition of correctness in LA.⊠

212.10.LEMMA.

If B is LA-compatible, Δ is LA-admissible for B and

$$B,\Delta \Vdash_{LA} \underline{c}(b_1,\dots,b_n)$$

for some $\underline{c}$ in Fl_n^{LA} and LA-terms $b_1,\dots,b_n$ $(n \geq 0)$ then

either

(1) $$B,\Delta \vdash_{LA} \underline{c}(b_1,\dots,b_n):\tau$$

or

(2) $$B,\Delta \vdash_{LA} \underline{c}(b_1,\dots,b_n):a:\tau$$

for some LA-term a.

Proof. By induction on correctness in LA.⊠

212.11.REMARK.

So, correct head-terms in LA can have only degrees 2 and 3.

As earlier, in 212.9. we can obtain some more information about correct head-terms in LA by a mere inspection of the $Esent_{\boxtimes}^{LA}$-rules. That is:

212.12.LEMMA.

Let B be LA-compatible and Δ be LA-admissible for B. If

$$B,\Delta \Vdash_{LA} \underline{c}$$

for $\underline{c}$ in Fl_0^{LA} then

(1) for some k in $Constr^{LA}$ such that

(11) $\underline{c} \equiv \underline{idf}(k)$ and

(12) $\Delta_0 \equiv \underline{ctx}(k)$

one has that

(2) k is in B

and, moreover,

(3) $$B,\Delta \vdash_{LA} \underline{c} : \underline{cat}(k).$$

Proof. As earlier, in each case (LA := PA,CA), by induction on correctness in LA.⊠

Now the general case will be completed by indicating the behaviour of correct head-terms a (in LA := PA,CA),where $\underline{\underline{head}}(a)$ is in Fl_n^{LA} with $n \geqslant 1$.

212.13.LEMMA.

Let B be LA-compatible and Δ be LA-admissible for B.If

$$B,\Delta \Vdash_{LA} \underline{c}(b_1,\ldots,b_n) \qquad (n \geqslant 1)$$

for some $\underline{c}$ in Fl_n^{LA} and LA-terms $b_1,\ldots,b_n$ $(n \geqslant 1)$ then

(1) for some k in $Constr^{LA}$ such that

(11) $\underline{c} \equiv \underline{\underline{idf}}(k)$

(12) $\underline{\underline{lh}}(\underline{\underline{ctx}}(k)) = n$

one has that

(2) k is in B

and,moreover,that

(3) $$B,\Delta \vdash_{LA} b_i : a_i[\![\bar{b} := \bar{v}]\!] \qquad (1 \leqslant i \leqslant n),$$

and

(4) $$B,\Delta \vdash_{LA} \underline{c}(b_1,\ldots,b_n) : a[\![\bar{b} := \bar{v}]\!],$$

where $(1 \leqslant i \leqslant n)$,

$$a_i \equiv \underline{\underline{pred}}(\underline{\underline{ass}}_i^n(\underline{\underline{ctx}}(k))),$$

$$a \equiv \underline{\underline{cat}}(k)$$

$$\bar{v} \equiv \langle v_1,\ldots,v_n \rangle = \underline{\underline{str}}_1(\underline{\underline{ctx}}(k)), \text{i.e.,}$$

$$v_i \equiv \underline{\underline{sub}}(\underline{\underline{ass}}_i^n(\underline{\underline{ctx}}(k)))$$

<u>Proof</u>.As ever,by induction on correctness in LA := PA,CA.(<u>Hint</u>: consider (Er-c).).∎

By now the "structural" characterization of PA-term-correctness was readily completed.

Before establishing a "structural" characterization of PA as a whole we go on insisting on the "structural" behaviour of correct CA-terms that are "properly in CA" (the terms <u>on</u> CA,i.e. the abstraction- and application-terms;for "on" see later on,section <u>3</u>,specifically <u>314</u>.)

First we consider application-terms in CA as regards their behaviour w.r.t. degrees,viz. we establish the admissible degrees for argument- and function-parts (consult 121.6.,etc. for terminology).

212.14.LEMMA.($\underline{\underline{arg}}$-degrees in CA).

If B is CA-compatible, Δ is CA-admissible and

$$B,\Delta \Vdash_{CA} \{a\}b$$

then for some CA-term a'

$$B,\Delta \vdash_{CA} a:a':\tau.$$

Proof.By induction on correctness in CA.⊠

212.15.REMARK.

Therefore,for any correct application-term in CA,a say,one has

$$\underline{\underline{dg}}(\underline{\underline{arg}}(a)) = 3.$$

This gives the following corollary.

212.16.COROLLARY.

If B is CA-compatible, Δ is CA-admissible for B and

$$B,\Delta \Vdash_{CA} \{a\}b$$

then

$$B,\Delta \Vdash_{CA} a.$$

Proof.By 212.14. and the definition of $Term_{⊠}^{CA}$.⊠

212.17.REMARK.

That is: for any CA-correct application-term a,the CA-term $\underline{\underline{arg}}$(a) is again CA-correct (relative to the same CA-compatible site,B say,and the same CA-admissible context for B).

212.18.($\underline{\underline{fun}}$-degrees in CA).

If B is CA-compatible, Δ is CA-admissible for B and

$$B,\Delta \Vdash_{CA} \{a\}b$$

then

$$B,\Delta \vdash_{CA} b:b':\tau,$$

for some CA-term b'.

Proof.By induction on correctness in CA.⊠

212.19.REMARK.

In other words, if a is a CA-correct application-term then

$$\underline{\underline{dg}}(\underline{\underline{fun}}(a)) = 3.$$

As earlier, we have function-part correctness as a corollary.

212.20.COROLLARY.

If B is CA-compatible, Δ is CA-admissible for B and

$$B,\Delta \Vdash_{CA} \{a\}b$$

then

$$B,\Delta \Vdash_{CA} b.$$

Proof. By 212.18. and the definition of $Term_{\boxtimes}^{CA}$.$\blacksquare$

212.21.REMARK.

So, for any correct application-term a in CA (relative to some LA-compatible B and some LA-admissible context for B) we also have that $\underline{\underline{fun}}(a)$ is correct in CA (relative to the same site and context).

It remains to consider abstraction-terms in $CorrTerm^{CA}$, as regards their behaviour w.r.t. degrees; specifically, to establish the admissible degrees for value- and domain-parts in CA (cf. 121.6.).

212.22.LEMMA. ($\underline{\underline{dom}}$-degrees in CA).

If B is CA-compatible, Δ is CA-admissible for B and

$$B,\Delta \Vdash_{CA} [v:a]b$$

then

$$B,\Delta \vdash_{CA} a:\tau,$$

i.e., one has always $\underline{\underline{dg}}(\underline{\underline{dom}}(a)) = 2$, for any abstraction term a in $CorrTerm^{CA}$.

Proof. As above, by induction on correctness in CA.$\blacksquare$

This gives domain-part correctness as a corollary.

212.23.COROLLARY.

If B is CA-compatible, Δ is LA-admissible for B and

$$B,\Delta \Vdash_{CA} [v:a]b$$

then

$$B,\Delta \Vdash_{CA} a,$$

i.e., for any abstraction-term a, if a is in $CorrTerm^{CA}$ then so is $\underline{\underline{dom}}(a)$.

Proof. By 212.22. and the definition of $Term_{⊠}^{CA}$.⊠

212.24. LEMMA. (Value-part correctness in CA.)

If B is CA-compatible, Δ is CA-admissible for B and

$$B,\Delta \Vdash_{CA} [v:a]b$$

then

(1) $\Delta' := \Delta[v:a]$ is CA-admissible for B and

(2) $B,\Delta' \Vdash_{CA} b$.

Proof. By induction on correctness in CA.⊠

212.25. LEMMA. (val-degrees in CA).

If B is CA-compatible, Δ is CA-admissible for B and a is an abstraction-term with

$$B,\Delta \Vdash_{CA} a$$

and

$$dg(B,\Delta,a) = n \qquad (n \geq 1)$$

then, where $a' = val(a), a'' = dom(a)$ and $\Delta' := \Delta[v:a'']$, for some v in Var such that v is fresh for Δ, one has also

$$dg(B,\Delta',a') = n.$$

Proof. By induction on correctness in CA.⊠

212.26. REMARK.

In particular, by 212.6.(4) one has, for any abstraction term a in $CorrTerm^{CA}$,

$$2 \leq dg(a) \leq 3,$$

so, by 212.25. and the motivation of the elliptic degree-notation in 212.3., we have also

$$2 \leq dg(val(a)) \leq 3.$$

We can now establish "structural" characterizations of application and abstraction terms in $CorrTerm^{CA}$, analogous to 212.9., 212.12. and 212.13. above. This will complete the "structural" characterization of CA-term-correctness.

The following Lemmas will collect information implicit in groups III and IV of correctness rules (for CA) listed in 211.

212.27. COMMENT.

We have seen in 212.15. and 212.19. that arg- and fun-degrees in CA have to be always "maximal", viz. 3. This will be also the case, in general, for abstraction-terms a in $CorrTerm^{CA}$, whose intended meaning is that $dg(a) = dg(val(a))$.

212.28.LEMMA.

Let B be a CA-compatible site and Δ be CA-admissible for B.If

$$B,\Delta \Vdash_{CA} \{a\}b$$

then there are CA-terms b_1,a' and b' such that

(1) $B,\Delta \vdash_{CA} b : b_1 : \tau$ (i.e.,$\underline{\underline{dg}}(b_1) = 2$)

(2) $B,\Delta \models b_1 \ \underline{\underline{conv}}_{CA} \ [v:a']b'$

(3) $B,\Delta \vdash_{CA} a : a' : \tau$ (i.e.,$\underline{\underline{dg}}(a') = 2$)

where if $\Delta' := \Delta[v:a]$ (with Δ' LA-admissible for B),then

(4) $B,\Delta' \vdash_{CA} b' : \tau$ (i.e.,$\underline{\underline{dg}}(b') = 2$)

and also

(5) $B,\Delta \vdash_{CA} \{a\}b : b'[\![a := v]\!]$.

with

(6) $\underline{\underline{dg}}(b) = \underline{\underline{dg}}(\{a\}b)$ (= 3).

Proof.By induction on the definition of correctness in CA.⊠

212.29.REMARK.

In Lemma 212.28. $\underline{\underline{conv}}_{CA}$ can be safely restricted to $\underline{\underline{conv}}_{\delta}^{\boxtimes}$ and even to $\underline{\underline{red}}_{\delta}^{\boxtimes}$ where the underlying atomic reductions can be always "external" (i.e.,"head-reductions",starting "outer-most/left-most" in a CA-term).

Reason: no application-term in CorrTerm^{CA} can have degree 2 (or 1).This excludes applications of ("external") atomic beta- and eta-reductions. Finally,"internal" applications of atomic reductions would do no harm, once we have some a' and some b' satisfying condition (2) in the statement of the Lemma.

212.30.COMMENT.

Lemma 212.28. appears implicitely in the "algorithmic" definition of AUT-68 of DE BRUIJN 77-52b (cf. also VAN DAALEN 80-73,V.4.2.).

Of course,in the statement of this Lemma bound occurrences of the variable v have to be chosen such as to avoid collisions (of bound variables).Cf. 123. above.

A similar analysis is also possible - in the case of CA - for correct abstraction terms.

212.31.LEMMA.

Let B be CA-compatible and Δ be CA-admissible for B.If

$$B,\Delta \;\Vdash_{CA}\; [v:a]\,b$$

then

(1) for some CA-term a',one has

(11) $B,\Delta' \;\vdash_{CA}\; v:a'$

(12) $B,\Delta \;\models_{CA}\; a' \;\underline{\underline{conv}}_{CA}\; a$

(13) $\underline{\underline{dg}}(a) = \underline{\underline{dg}}(a') = 2,$

(2) $\Delta' := \Delta[v:a]$ is CA-admissible for B,

(3) $B,\Delta' \;\Vdash_{CA}\; b,$ with $2 \leq \underline{\underline{dg}}(b) \leq 3,$

(4) $\underline{\underline{dg}}([v:a]\,b) = \underline{\underline{dg}}(b).$

Moreover,if

$$B,\Delta' \;\vdash_{CA}\; b:b'$$

then

(5) $B,\Delta \;\vdash_{CA}\; [v:a]\,b : b'',$

where

$$b'' := \begin{cases} [v:a']\,b', & \text{if } \underline{\underline{dg}}(b) = 3 \\ b' \;(\equiv \tau), & \text{if } \underline{\underline{dg}}(b) = 2. \end{cases}$$

Proof.By induction on the definition of correctness in CA,using facts proved earlier (on degrees).⊠

212.32.REMARK.

In the statement of 212.31. the bound occurrences of the variable v have to be chosen such as to avoid collisions of bound variables. (See the conventions on alpha-conversion in 123. above.)

212.33.COMMENT.

Lemma 212.31. is,implicitely,part of the "algorithmic" definition of AUT-68 appearing in DE BRUIJN 77-52b (referred to earlier).

212.34.COMMENT.

It is now easy to see that the preceeding "structural" Lemmas 212.9.,212.12. 212.13. (for LA := PA,CA) and 212.28.,212.31. (for LA := CA),together with the $Term_{\alpha}^{LA}$-initialization rule (Ti) (for LA := PA,CA),can be strengthenend up to the corresponding equivalences.

They insure,jointly,the (epi-theoretic) correctness of inductive proofs on the "structure" of a LA-term in $CorrTerm^{LA}$ (LA := PA,CA) (whence our qualification of the Lemmas above).

Moreover,the named Lemmas provide the admissible "predicates" (pred-parts) for correct Esentences in LA (in $CorrEsent^{LA}$,i.e.,they furnish the admissible "typings" for LA-terms in $CorrTerm^{LA}$,where LA := PA,CA).

The latter kind of information was actually embodied - in the usual,more or less "algorithmic" formulations of PAL(-THE) and AUT-68 - in the action of a specific epifunction on "correct expressions" (roughly corresponding to our terms in $CorrTerm^{LA}$),called "mechanical typing function".(This is usually denoted by "CAT" or "typ","cantyp";cf.ZANDLEVEN 77-36,VAN DAALEN 73-35,6.4.2.3., DE BRUIJN 77-52b,JUTTING 79-46,4.1.0. or Appendix 9.,as well as the description of typ and cantyp - relative to "correct books" and "correct contexts" - in VAN DAALEN 80-73,IV.3.2.3. et sq.,V.3.2.4.,etc.).

Accurately,this is a recursive epifunction totally defined on $CorrTerm^{LA} - \{\tau\}$ with values in $CorrTerm^{LA}$ (here: LA := PA,CA) and also depends on LA-compatible sites and LA-admissible contexts.

213. Basic epitheory: invariance under site- and/or context-expansion.

We shall list here several more or less trivial but useful lemmas concerning the "invariance of correctness" in syntagmatic categories of LA (:= PA,CA) under "LA-compatibility and/or LA-admissibility preserving expansions" in the corresponding correctness categories (i.e., $Site^{LA}_{\boxtimes}$ and/or $Contx^{LA}_{\boxtimes}$).

Specifically, we establish that "expanding" correctness categories like $Site^{LA}_{\boxtimes}$ and/or $Contx^{LA}_{\boxtimes}$ preserves correctness for the remaining syntactic categories.

"Expansion" in (LA-compatible) sites will simply mean "taking super-sets" within $Site^{LA}_{\boxtimes}$, whereas "expansion" of (LA-compatible) contexts will be understood - roughly speaking - as "increasing contexts-length" within $CorrContx^{LA}$.

We examine first the case of contexts. As earlier, LA := PA,CA if not otherwise specified.

213.1. LEMMA. (Lexical variants for contexts.)

Let B be an LA-compatible site and $\Delta := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ $(n \geqslant 1)$ be an LA-admissible context for B.

Where $v'_1,\ldots,v'_n$ are pairwise distinct (i.e., $\bar{v}' := (v'_1,\ldots,v'_n)$ is in $Svar_n$ and each v'_i $(1 \leqslant i \leqslant n)$ is fresh for B) with also

$$a'_i := a_i [\![\bar{v}'_i =: \bar{v}_i]\!]$$

the LA-context

$$\Delta' := \Delta_0[v'_1:a'_1]\ldots[v'_n:a'_n]$$

is LA-admissible for B.

Proof. Easy induction on correctness in LA. $\boxtimes$

213.2. THEOREM. (Correctness-invariance under lexical variants of contexts.)

Let B be LA-compatible and Δ be LA-admissible for B. Where v'_i, a'_i, Δ' are as in the statement of Lemma 213.1. and, moreover, for LA-terms a,b

$$a' := a [\![\bar{v}' := \bar{v}]\!]$$
$$b' := b [\![\bar{v}' := \bar{v}]\!]$$

we have:

(1) If $B,\Delta \vdash_{LA} a : b$ then $B,\Delta' \vdash_{LA} a' : b'$,

(2) If $B,\Delta \Vdash_{LA} a$ then $B,\Delta' \Vdash_{LA} a'$

and

(3) If $B,\Delta \models_{LA} a \underline{\underline{R}} b$ then $B,\Delta' \models_{LA} a' \underline{\underline{R}} b'$,

where $\underline{\underline{R}} := \underline{\underline{contr}}_{LA}, \underline{\underline{conv}}^1_{LA}, \underline{\underline{red}}_{LA}, \underline{\underline{conv}}_{LA}$.

Proof. By induction on correctness in LA, using 213.1.$\boxtimes$

This gives the following bunch of consequences (cf. 15. for $\underline{\underline{\text{sub}}}$).

213.3. THEOREM. (Correctness-invariance under context-expansion.)

Let B be an LA-compatible site and Δ, Δ' be LA-admissible contexts for B. Let $\Delta \ \underline{\underline{\text{sub}}}\ \Delta'$ (that is: Δ is a subcontext of Δ').

(1) If $B, \Delta \vdash_{LA} a:b$ then $B, \Delta' \vdash_{LA} a:b$.

(2) If $B, \Delta \Vdash_{LA} a$ then $B, \Delta' \Vdash_{LA} a$.

(3) If $B, \Delta \models_{LA} a \ \underline{\underline{R}}\ b$ then $B, \Delta' \models_{LA} a \ \underline{\underline{R}}\ b$,

where $\underline{\underline{R}} := \underline{\underline{\text{contr}}}_{LA}, \underline{\underline{\text{conv}}}^{1}_{LA}, \underline{\underline{\text{red}}}_{LA}, \underline{\underline{\text{conv}}}_{LA}$.

Proof. (1) By induction on the "derivation" of $B, \Delta \vdash_{LA} a:b$ in LA. (2) By (1) and the definition of $\text{Term}^{LA}_{\boxtimes}$. (3) From (2) and the definition of $\underline{\underline{R}}^{\boxtimes}$ (i.e., $\underline{\underline{R}}$ restricted to $\text{CorrTerm}^{LA} \times \text{CorrTerm}^{LA}_{\boxtimes}$.). $\boxtimes$

213.4. COMMENT.

Theorem 213.3. is the analogue of van Daalen's Weakening Theorem (for "correct/admissible" contexts) in VAN DAALEN 80-73, V.2.9.3.

The "converse" statements (corresponding to 213.3.(1)-(3) above) were called "Strengthening Rule(s)" in VAN DAALEN 80-73., V.2.6. and were taken as primitive rules in some "reference"-formulations of the AUT-languages studied there. The corresponding statements for LA := PA, CA are as follows:

Let B be an LA-compatible site and Δ, Δ' be LA-admissible contexts for B, with $\Delta' \ \underline{\underline{\text{sub}}}\ \Delta$, where $\Delta' := \Delta_0[v_1:a_1] \ldots [v_n:a_n]$, and a,b in Term^{LA} such that: whenever v is in $FV(a) \sqcup FV(b)$ one has also $v \equiv v_i$, for some $i, 1 \leqslant i \leqslant n$.

(1) If $B, \Delta \vdash_{LA} a:b$ then $B, \Delta' \vdash_{LA} a:b$,

(2) If $B, \Delta \Vdash_{LA} a$ then $B, \Delta' \Vdash_{LA} a$,

and, where $\underline{\underline{R}} := \underline{\underline{\text{contr}}}_{LA}, \underline{\underline{\text{conv}}}^{1}_{LA}, \underline{\underline{\text{red}}}_{LA}, \underline{\underline{\text{conv}}}_{LA}$, as earlier,

(3) if $B, \Delta \models_{LA} a \ \underline{\underline{R}}\ b$ then $B, \Delta' \models_{LA} a \ \underline{\underline{R}}\ b$.

Using (1)-(2) above qua rules in the primitive correctness part of LA (:= PA, CA) allows to "simplify" LA-admissible contexts, from the very beginning, up to "minimal non-redundant" (LA-admissible) contexts for given correct LA-terms, resp. correct $\underline{E}$-sentences in LA. (Indeed, the "strengthening rule" (3) would follow from (2) and the definition of $\underline{\underline{R}}$ in our formulations above.)

In VAN DAALEN 80-73,the corresponding "Strengthening Rules" have been employed only as "technical rules" in order to simplify proofs involving the "extensional" versions of AUT-68,AUT-QE,etc. and the argument that they are actually dispensable is rather tedious.

213.5.COMMENT.

As expected,Theorem 213.3. will allow to "modify" LA-admissible contexts along a given correctness proof,by additioning "redundant" assumptions (within the limits of CorrContx^{LA},of course).It says,in the end,that any such an LA-admissible "expansion" should preserve correctness for any one of the "bigger" categories depending on Contx$^{LA}_{\boxtimes}$.
In particular,this result allows us to pass from some "local environment" in some LA-compatible site to a more comprehensive one,involving the former. Something analogous is also possible for sites.

213.6.LEMMA.(Site$^{LA}_{\boxtimes}$-expansion for sound LA-constructions.)

Let B,B' be LA-compatible.If B is a subset of B' and k is sound in/for B then k is also sound in/for B'.

Proof.Trivial,by the definition of soundness of LA-constructions in LA-compatible sites.⊠

213.7.THEOREM.(Site$^{LA}_{\boxtimes}$-expansion for LA-contexts,$\underline{E}$-sentences and LA-terms.)

Let B,B' be LA-compatible sites such that B is a subset of B' and let Δ be an LA-admissible context for B.Then (for all LA-terms a,b):

(1) Δ is LA-admissible for B'.

(2) If $B,\Delta \vdash_{LA} a:b$ then also $B',\Delta \vdash_{LA} a:b$.

(3) If $B,\Delta \Vdash_{LA} a$ then also $B',\Delta \Vdash_{LA} a$.

Proof.By induction on correctness in LA (simultaneously).⊠

This gives the following consequence(s).

213.8.COROLLARY.(Site$^{LA}_{\boxtimes}$-expansion for correct $\underline{C}$-,$\underline{R}$- and $\underline{Q}$-sentences.)

Let B,B' be LA-compatible sites.If B is a subset of B',Δ is LA-admissible for B and

$$B,\Delta \models_{LA} a \;\underline{\underline{R}}\; b$$

then also

$$B',\Delta \models_{LA} a \;\underline{\underline{R}}\; b$$

(where $\underline{\underline{R}}$:= $\underline{\underline{\text{contr}}}_{LA}$,$\underline{\underline{\text{red}}}_{LA}$,$\underline{\underline{\text{conv}}}_{LA}$).

Proof. By 213.7.(3) and the definition of $\underline{\underline{R}}$.⊠

Collecting the facts on "expansions" we have the following Theorem.

213.9. THEOREM.

Let B,B' be LA-compatible sites and Δ,Δ' be LA-contexts such that

(i) B is a subset of B',

(ii) Δ is a subcontext of Δ' (i.e., $\Delta\ \underline{\underline{sub}}\ \Delta'$) and

(iii) Δ,Δ' are LA-admissible for B.

Then the following implications hold:

(1) if $B,\Delta \vdash_{LA} a:b$ then $B',\Delta' \vdash_{LA} a:b$,

(2) if $B,\Delta \Vdash_{LA} a$ then $B',\Delta' \Vdash_{LA} a$,

(3) if $B,\Delta \models_{LA} a\ \underline{\underline{R}}\ b$ then $B',\Delta' \models_{LA} a\ \underline{\underline{R}}\ b$,

where $\underline{\underline{R}} := \underline{\underline{contr}}_{LA}, \underline{\underline{red}}_{LA}, \underline{\underline{conv}}_{LA}$.

Proof. (1) Then, by 213.7.(1), Δ' is LA-admissible for B', while, by 213.7.(2),

$$B',\Delta \vdash_{LA} a:b.$$

Hence, by 213.3.(1), one has the desired result.

(2) By 213.7.(1),(3) and 213.3.(2). Finally, (3) follows by 213.7.(1), 213.8. and 213.3.(3).⊠

213.10. COMMENT.

The moral of 213.9. is as expected: the addition of "redundant" (B-correct) LA-constructions to some LA-compatible site B and (the addition of) "redundant" assumptions within some "correct" LA-context (such as to preserve LA-admissibility) do not alter the corresponding concepts of correctness for LA-formulas and LA-terms (where "LA-formulas" are to be taken in the extended sense: either E-sentences or C- or R- or Q-sentences).

214. Substitution and category correctness.

We are now able to prove some more useful Theorems on the behaviour of LA-terms in $\mathrm{CorrTerm}^{LA}$ (LA := PA,CA).

Specifically, we will be first concerned with deriving several important facts on (simultaneous) substitutions in $\mathrm{CorrTerm}^{LA}$.

214.1. THEOREM. (Simultaneous substitution for E-sentences.)
Let B be an LA-compatible site, $\Delta := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ and Δ' be LA-admissible contexts for B. If

$$B,\Delta' \vdash_{LA} b_i : a_i[\![\bar{b} := \bar{v}]\!] \qquad \text{for all } i, 1\leq i\leq n,$$

and

$$B,\Delta \vdash_{LA} a : a'$$

then

$$B,\Delta' \vdash_{LA} a[\![\bar{b} := \bar{v}]\!] : a'[\![\bar{b} := \bar{v}]\!].$$

Proof. By induction on the "derivation" of

$$B,\Delta \vdash_{LA} a : a'$$

in LA. (Cf. VAN DAALEN 80-73, V.2.9.4. for the analogous result in "reference" AUTOMATH.). ⊠

Similarly, we have the following "elliptic" counterpart of 214.1., for correct LA-terms.

214.2. THEOREM. (Simultaneous substitution for LA-terms.)
Let B be an LA-compatible site, $\Delta := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$ and Δ' be LA-admissible contexts for B. If

$$B,\Delta' \vdash_{LA} b_i : a_i[\![\bar{b} := \bar{v}]\!] \qquad \text{for all } i, 1\leq i\leq n,$$

and

$$B,\Delta \Vdash_{LA} a$$

then

$$B,\Delta' \Vdash_{LA} a[\![\bar{b} := \bar{v}]\!].$$

Proof. As earlier, for 212.1., using induction on the derivation of $B,\Delta \Vdash_{LA} a$. ⊠

The above give immediately the particular cases involving "single substitutions".

214.3.COROLLARY.("Single substitution theorem".)

Let B be an LA-compatible site and Δ be some LA-admissible context for B. Assume that

$$B,\Delta \vdash_{LA} b : a.$$

Then

(1) if $\quad B,\Delta_0[v:a] \vdash_{LA} a' : a''$

then

$$B,\Delta \vdash_{LA} a'[\![b := v]\!] : a''[\![b := v]\!]$$

and, similarly,

(2) if $\quad B,\Delta_0[v:a] \Vdash_{LA} a'$

then

$$B,\Delta \Vdash_{LA} a'[\![b := v]\!].$$

Proof.(1) From 213.1.(2) From 213.2.⊠

214.4.REMARK.(Simultaneous substitutions for C-,R- and Q-sentences.)

Once we have proved 214.2. we have the appropriate versions of the (simultaneous/"single") Substitution Theorem for C-,R- and Q-sentences (for these work as mere abbreviations in our formulations above).

That is, with $\underline{\underline{R}}$ for any one of $\underline{\underline{contr}}_{LA}$, $\underline{\underline{red}}_{LA}$, $\underline{\underline{conv}}_{LA}$ we obtain the following:

If B is LA-compatible and Δ,Δ' are LA-admissible for B (where Δ is as in Theorem 214.1., i.e., $:= \Delta_0[v_1:a_1]\ldots[v_n:a_n]$) and

$$B,\Delta' \vdash_{LA} b_i : a_i[\![\bar{b} := \bar{v}]\!], \qquad (1 \leqslant i \leqslant n),$$

$$B,\Delta \models_{LA} a \ \underline{\underline{R}}\ a'$$

then

$$B,\Delta' \models_{LA} a[\![\bar{b} := \bar{v}]\!] \ \underline{\underline{R}}\ a'[\![\bar{b} := \bar{v}]\!].$$

(This holds also for $\underline{\underline{R}} := \underline{\underline{exp}}_{LA}$, i.e., for the converse of $\underline{\underline{contr}}_{LA}$ and, consequently, for $\underline{\underline{conv}}^1_{LA}$, the "atomic convertibility" relation, as expected.)

Of course, if n = 1, the corresponding statements would concern the "single substitution" case.

In the statement of the primitive correctness rules for LA (:= PA,CA) the rules having an output in $Esent^{LA}_{⊠}$ (i.e., something of the form $B,\Delta \vdash_{LA} a:b$, with B in $Site_{⊠}$ and Δ in $CorrContx^{LA}_{⊠LA}$), no stipulation was made on the correctness of the "predicates" b in "a:b". (Correctness for subject-parts follows from (Tr), of course.). This will be obtained now as a metatheorem.

214.5.THEOREM.("Correctness for categories".)

Let B be LA-compatible and Δ be LA-admissible for B.If

$$B,\Delta \ \vdash_{LA} \ a{:}b$$

then

$$B,\Delta \ \Vdash_{LA} \ b.$$

Proof.By induction on the "derivation" of $B,\Delta \vdash_{LA} a{:}b$ using (Ti),213.3.,214.1. and facts proved in 212. above.⊠

214.6.COROLLARY.

Let B be LA-compatible and Δ be LA-admissible for B.If

$$B,\Delta \ \vdash_{LA} \ a{:}b$$

then

$$B,\Delta \ \Vdash_{LA} \ a,b.$$

Proof.By (Tr) and 214.5. above.⊠

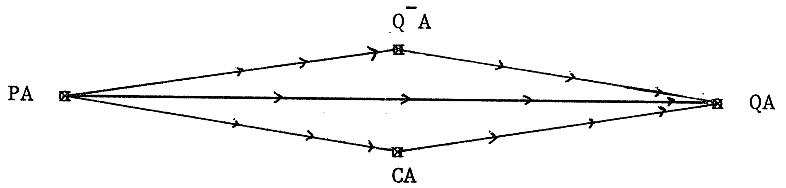
22. Correctness for $Q^{-}A$ and QA.

This section is devoted to the study of correctness in two extensions of CA which we call here $Q^{-}A$ and QA.

Accurately, QA is supposed to be the abstract version of the official "reference" AUT-QE language, whereas $Q^{-}A$ is a sub-language of QA, intended to "mimick", on the pattern of the present approach, the abstract structure of de Bruijn's AUT-QE-NTI (that is: AUT-QE-Without-Type-Inclusion; cf., e.g., DE BRUIJN 78-56).
Prima facie, the difference between QA and $Q^{-}A$ consists of the absence of the Rule of Category Inclusion (cf. below) which is characteristic for QA and does not hold in/for $Q^{-}A$. (This is also, roughly speaking, the difference between AUT-QE and AUT-QE-NTI, where the latter lacks the analogous rule of Type Inclusion).

Both $Q^{-}A$ and QA are extensions of PA, in the same sense CA was. Moreover, QA is a (rule-)extension of CA, too (in the sense the primitive correctness rules of CA should hold in/for QA), but this is not the case for $Q^{-}A$ (so the Rule of Category Inclusion is essential in "deriving" the primitive rules of CA which are absent from - the present formulation of the correctness rules of - QA).

The situation described above can be easily depicted by the following diagram of all possible inclusions (we ignore, for the moment being, any distinction of "extensionality type"; cf. below). Here "$LA_1 \twoheadrightarrow LA_2$" stands for "$LA_1$ is a sub-language of LA_2" or "LA_2 is an extension of LA_1", where the arrow points out to the "bigger" language:



Modulo the convention in 10.4. above (concerning the elimination of the *proof-type symbol* from the alphabet of any LA of concern here), the "free" parts of CA, $Q^{-}A$ and QA coincide (in the sense they have the same set of terms, *E*-sentences, etc.), whereas PA is a sub-language of any of the remaining (abstract) languages also inasfar its "free" part is concerned (it lacks abstraction- and application-terms).
So the arrows of the preceeding diagram are to be interpreted *uniformly* as depicting *correctness-rule-extensions* (see, however, the Comment following below for some more subtleties).

So we will display the appropriate sets of correctness rules Rl_{Q^-A} and Rl_{QA}, thereby completing Definition 20.1. for the cases LA := Q^-A,QA.

Rigorously,in each case,we are defining two distinct abstract AUT-languages LA by the same set of correctness rules Rl_{LA} (as earlier in 21.),viz. β-LA and βη-LA,according to the "extensionality type" taken as basic in the associated reduction system LA.Still,for most of our (syntactic) purposes the correspon-ding distinguo won't matter.

22.1.COMMENT.("Derivable" vs "admissible rules".)

In our informal discussion above,(epi-theoretic) concepts like "rule-extension" and "extension" were used rather freely in connection with the abstract "langua-ges" LA (:= PA,CA,Q^-A,QA).As some misunderstandings are not quite unlikely the matter deserves some comment.

Following N.G.de Bruijn and standard meta-theoretic terminology used currently in connection with the AUTOMATH-project (at least in Eindhoven),we did and will consequently speak about AUTOMATH-languages (whether in "reference" or in abstract format) rather that about AUTOMATH-proof-systems or AUTOMATH-(deduction-)systems.That is: the correctness part of an (abstract) AUTOMATH language is to be actually taken as a part of the language,as well as its "free" /"canonical" syntactic part.

This meta-linguistic habits may be somewhat mis-leading (at least for some logicians): indeed,in a more rigorous meta-theoretic setting - which,by the way, won't be too relevant in connection with AUTOMATH - one would always want to distinguish between a bare "language" and some super-imposed "proof-system" "system of derivations" on/for this "language".In the latter acception,any AUTOMATH-language (whether "abstract" or not) should be,of course,a proof-system (accurately: a proof-checking system and not a proof-producing device, say not a "system for proving theorems" [1]).

---

1) N.G.de Bruijn et al. have repeatedly insisted on the fact any language in the AUTOMATH-family is neither intended nor suited to be a theorem-proving system (cf.DE BRUIJN 73-34,80-72,21;VAN DAALEN 80-73,I.1.9.,etc.;however see VAN DAALEN op.cit.,I.2.3. and the suggestion to use "attachments" to an AUT-verifier in DE BRUIJN 80-72,21).The point is oft missed and "remarks" like: "one disconcerting aspect of the AUTOMATH language is ... the fact that it seems not to make any use of the improved automatic theorem proving techniques developed during the last decade" (BROWN 81) reveal a mere misunderstanding of the nature of AUTOMATH.

Thus,as a system of (correctness) rules,an AUT-language is,roughly,comparable with any "deductive system" (a natural deduction system say;actually,T-versions of some "reference" AUT-languages - where "T-" labels the "natural deduction style" as in CURRY 63,etc.- have been proposed and worked out so far in the existing literature on AUTOMATH;cf.,e.g.,van Daalen's Master's Thesis: VAN DAALEN 72-28,I.5. or the informal treatement of AUT-68 in VAN DAALEN 80-73, I.,4.5. and the general discussion of the subject in NEDERPELT 77-54,etc.). In this respect an (abstract) AUT-language is a production system,viz. its (primitive) correctness rules are supposed to generate (uniquely) the family of correctness categories of the language (where the latter are either sets of words over the alphabet of the language or finite sets of sets of words - in our present approach).

So far we have been speaking only of "primitive" rules,in any such a production system,while any "derived" rule of the system has been viewed - somewhat uncommitally - as a "meta-theorem" about/on the system.One should perhaps insist more on the possible meaning of "derived" here.Specifically,we did not pay attention too much in the (epi-theoretic) proofs (on LA's) above to the "proof-theoretic strength" of the apparatus needed to obtain "derived" rules (in LA's). Indeed,a correctness rule may be called a "derived rule" in at least two distinct acceptions (the distinguo is very likely due to LORENZEN 55,if not to H.B.Curry;we hereafter use Lorenzen's terminology,but only with local import). Let Rl_0 be a set of primitive rules for some (production system) LA.Then the set of derivable rules over Rl_0 (resp. in LA) is the least set Rl,containing Rl_0 and which is closed under

(1) reflexivity (i.e., $X_1,\ldots,X_n \vdash X_i$ is in Rl;where the turnstile is to be taken in the usual Rosser-Kleene sense;cf.REZUS 81,etc.)
(2) addition of an arbitrary premiss (i.e.,(K) in CURRY 63;there "weakening")
(3) contraction of premisses,(i.e.,(W) in CURRY 63),
(4) permutation of premisses,(i.e.,(C) in CURRY 63),
(5) transitivity (i.e.,some form of "cut";Curry's ($\vdash_{\boxtimes}$) say,or (chain) in REZUS 81).

(The "closure"-conditions are,obviously,mere variants of the "structural" rules in some Gentzen-style sequent formulation of first-order logic say.Historically, this concept of derivability for rules is explicitely due to J.B.Rosser,whence "Rosser-derivability" in REZUS 81;but apparently earlier implicit manipulations of it - in a less general setting - can be found already in Gentzen's writings.)

It is clear that - in the acception above - only very few rules are "derivable" in LA (:= PA,CA say),beyond those that are already "primitive" in LA (i.e.,with our standard notation: in Rl_{LA}).
There is still another sense in which a rule can be said to be "derived" in/for some (abstract) AUT-language LA,viz. whenever it is "admissible" in the acception of Curry and Lorenzen.
Roughly speaking and with application to (the production systems) LA as introduced here,we say that a rule is (Curry-Lorenzen-) admissible in/for some LA if it does not increase some correctness category of LA when added to the primitive apparatus of correctness in LA.In particular,any derivable rule in LA is also admissible in LA but not conversely,for in the case of "admissibility" nothing is said about the proof-theoretic strength of the methods allowed to obtain some "admissible rule" for LA.
Since the former sense of "derived" (viz. "derivability") is rather poor in content we may eventually need using "derived" in the larger acception (of Curry and Lorenzen) when relativized to the case of rules in LA.(In VAN DAALEN 80-73,no similar distinction is made concerning the rules of "reference" AUT-languages therein discussed,but by a mere inspection of the complexity of some proofs "by induction on correctness",it should be clear that in most of the cases the larger acception is meant by "derived rule".)
This distinction relativizes directly to "rule-extensions";namely,LA_1 is a rule-extension of LA_2 modulo derivability if the set of derivable rules in LA_1 is a super-set of the set of derivable rules in LA_2,whereas LA_1 is said to be a rule-extension of LA_2 modulo admissibility if the set of admissible rules (à la Curry-Lorenzen) of LA_1 is a super-set of the set of admissible rules of LA_2.
Note also that - in the picture displayed earlier - "correctness-rule-extension" should be rather taken modulo derivability.(This is,of course,at least in the case of the abstract AUT-languages of concern here,a matter of choice of the corresponding sets of primitive rules: Rl_{LA},in our notation.In our present description the sets Rl_{LA} - for LA := PA,CA,$Q^{-}A$,QA etc. - are chosen such as to be minimal for the corresponding formulation of LA;in other words: no primitive rule in any of the sets Rl_{LA} seems to be dispensable.Cf. below.)

22.2.REMARK.

As earlier for PA and CA,the "language definition" for $Q^{-}A$ and QA in this section corresponds to the "E-definition" of the corresponding "reference" AUT-languages in VAN DAALEN 80-73.

22.3.COMMENT.

In detail,the "structure" of the sets Rl_{LA} (LA := Q^-A,QA) is as earlier for CA,with some minor modifications.

First,

(1) the structural rules,concerning the compatibility of sites and the admissibility of contexts in LA (:= Q^-A,QA),

(2) the basic rules,concerning the correctness of E-sentences and the correctness of LA-terms not involving abstractions or applications

as well as

(3) the rules of category conversion for LA (atomic reduction/expansion)

remain unchanged for LA := Q^-A,QA.

Besides these,Rl_{Q^-A} and Rl_{QA} contain also

(5) specific rules,concerning

(51) the correctness of LA-terms (on LA := Q^-A,QA;for "on" see 314.), involving abstractions (and applications): viz. (Tr-QA) and

(52) the correctness of E-sentences (on LA := Q^-A,QA),involving abstractions and applications,viz.

(521) application rules: (app-1),as in CA,and (app-2),and

(522) abstraction rules: (abs-1-QA) and - as in CA - (abs-2).

Moreover,Rl_{QA} contains a new rule (absent from Rl_{Q^-A},too),viz.

(6) the rule of category inclusion:(CI).

So,in order to prove that Q^-A (and therefore QA) is a "correctness-rule-extension" of CA we should "derive" the specific CA-rule (abs-1-CA).

It will be seen that the restricted sense of "derived" will suffice for this (cf. 22.1. above,for "derivability").

Rigorously,the present formulations are non-redundant when the rules in Rl_{Q^-A} and Rl_{QA} are taken as "atomic units" (i.e.,if we do not allow "decompositions" in "instances" or "partial cases").It is not so after a detailed "degree-analysis",similar in nature with that performed in 212. for PA and CA.Indeed,it will be seen easily that the instances of the "specific" $Term_{\boxtimes}^{LA}$-recursion Rule (Tr-QA) (LA := Q^-A,QA) which make sense for CA are already "derivable" in CA (in the technical sense of "derivable" noted earlier).

This - and some other minor redundancies - are due to the fact we did not want to include degree-considerations within the language definition from the very beginning,but prefer to obtain them as meta-theorems.

221. Correctness rules for $Q^{-}A$ and QA.

For the correctness rules in Groups I through III the reader should consult 211. (This is also the case for (app-1) and (abs-2) which appear in Group IV in 211.)

Now the definition of $Q^{-}A$- and QA-(correctness) will be completed by listing the remaining rules in $Rl_{Q^{-}A}$ (and Rl_{QA}) which are not in Rl_{CA}.

V. SPECIFIC RULES.

$LA := Q^{-}A, QA.$

V.1. $Term_{\alpha}^{LA}$-RECURSION RULE.

(Tr-QA) If B is LA-compatible,

Δ_n is LA-admissible for B,

$B, \Delta_n \vdash_{LA} a:\tau$ and

$B, \Delta_{n+1}^{a} \Vdash_{LA} b$

then

$B, \Delta_n \Vdash_{LA} [v:a]b$,

where $\Delta_n := \Delta_0 [v_1:a_1] \ldots [v_n:a_n]$,

v in Var, v fresh for Δ_n, v not in FV(a) and

$\Delta_{n+1}^{a} := \Delta_n [v:a]$.

V.2. $Esent_{\alpha}^{LA}$-RECURSION RULES.

V.2.1. APPLICATION RULES.

V.2.1.1. APPLICATION RULE 1. (app-1) : as for CA (cf. 211. rule IV.1.).

V.2.1.2. APPLICATION RULE 2.

(app-2) If B is LA-compatible,

Δ_n is LA-admissible for B,

$B, \Delta_n \vdash_{LA} a':a:\tau$

$B, \Delta_n \vdash_{LA} b':b:[v:a]a''$

then

$B, \Delta_n \vdash_{LA} \{a'\}b' : \{a'\}b$.

V.2.2.ABSTRACTION RULES.

V.2.2.1.ABSTRACTION RULE 1 FOR $Q^{-}A$,QA.

Let $T := [v_1':a_1']\ldots[v_m':a_m']\tau$, $(m\geqslant 0)$.

(abs-1-QA) If B is LA-compatible,

Δ_n is LA-admissible for B,

$B,\Delta_n \vdash_{LA} a:\tau$

$\Delta^a_{n+1} := \Delta_n[v_{n+1}:a]$ is LA-admissible for B,

$B,\Delta^a_{n+1} \vdash_{LA} b : T$

then

$B,\Delta_n \vdash_{LA} [v_{n+1}:a]b : [v_{n+1}:a]T.$

V.2.2.2.ABSTRACTION RULE 2. (abs-2) : as for CA (cf. <u>211</u>. rule IV.2.2.)

VI.CATEGORY INCLUSION RULE

(CI) If B is QA-compatible,

Δ_n is QA-admissible for B and

$B,\Delta_n \vdash_{QA} b: [v_1':a_1']\ldots[v_m':a_m'][v':a]\tau$, $(m\geqslant 0)$,

then

$B,\Delta_n \vdash_{QA} b: [v_1':a_1']\ldots[v_m':a_m']\tau$

221.1.COMMENT.

This completes the definition of $Q^{-}A$ and QA (that is 20.1.,for the cases in point).To sum up,$Rl_{Q^{-}A}$ contains the rules listed under 211.1.(I.- III.) and moreover:

V.(Tr-QA),(app-1),(app-2),(abs-1-QA),(abs-2),

whereas Rl_{QA} has also

VI. (CI)

besides the above.

The classification of the correctness rules according to the (correctness-) categories of their outputs (cf. 211.1.) is,again,preserved for $Q^{-}A$,QA.

As earlier for PA and CA,the present cases of 20.1. are <u>inductive definitions</u> where the underlying recursion runs by <u>simultaneous induction</u> as intended.

The "initialization"-clauses remain for $Q^{-}A$ and QA as they were earlier for CA,and obviously,only the corresponding "recursion"-clauses are modified.

221.2.COMMENT.

Note also that (CI),the Rule of Category Inclusion for QA,is a "modifying" rule;viz. it "transfers" information from old QA-compatible sites to new ones but also "modifies" this information in some sense different from what the rules of Category Conversion do.In general,the resulting "modification" amounts to a "loss of information" and it is,in some sense,"definitive": the information "lost" by some application of (CI) cannot be "recovered" at later stages any more.

This rule has also some ad hoc character which is,in general,absent from the remaining rules of QA.

In particular,(CI) - or better,its "reference"-analogue,called "type-inclusion",in AUT-QE - is well-known for its unpleasant (if not disastruous) effects: for instance,it distroys nice properties like the Unicity of Categories ("types"),otherwise holding for languages without (CI) - see VAN DAALEN 80-73.

Quite a lot of work has been spent on the possible approaches to circumvent (CI),still preserving the proper useful (and nice) features of QA (resp. AUT-QE).In this sense,the presence of "type-inclusion" in AUT-QE was at least fertile for it gave rise to a number of "reference"-AUT rivals of the languages initially devised and implemented in Eindhoven (TH).

As pointed out by N.G.de Bruijn (cf. DE BRUIJN 78-56,etc.) the effect of "type-inclusion" (and therefore that of our (CI) above) can be "mimicked" already in AUT-QE-NTI (and hence in $Q^{-}A$) once we decide to write only correct "books" that are proper extensions of a given fixed "book" containing something similar to the axioms of universal quantification.(In an abstract setting: consider only $Q^{-}A$-compatible sites containing a fixed $Q^{-}A$-site $B_{\underline{all}}$ say as a subset,where $B_{\underline{all}}$ contains at least some sound p-construction for "all" and sound p-constructions for "all-introduction" and "all-elimination".).Not too much has been done in this direction so far (as regards implementation say).Still this solution is by no means less ad hoc than the presence of Category Inclusion ("type-inclusion") in AUT-QE.

An alternative way-out (due to Jeffery Zucker) consists of strengthening both the "free"-part of the language and its correctness-part by addition of new constructors;this did eventually lead to a very strong AUT-language ("reference"),viz. AUT-Pi (whose abstract version is not studied here).However the "official" formulation of AUT-Pi does not contain all instances of "type-inclusion" as "derived rules" (even in the Curry-Lorenzen sense). (See,e.g.,DE BRUIJN 77-51,78-56 and VAN DAALEN 80-73,VIII.6.1.,for details.) An informal discussion of AUT-Pi can be found in ZUCKER 75-42,while formal details can be recovered from VAN DAALEN 80-73.,VIII.

222. Basic epi-theory for $Q^{-}A$ and QA.

The basic properties of the correctness categories of PA and CA - discussed in 212. through 214. above - will be seen to transfer easily, under minor changes, to the "bigger" languages $Q^{-}A$ and QA.
We shall insist below only on the main differences, induced by the presence of the specific $Q^{-}A$- and QA-correctness rules.
Hereafter, if not otherwise specified (explicitely), LA := $Q^{-}A$ or QA.

222.1. REMARK.

The sets $CorrContx^{LA}$, $CorrConstr^{LA}$, $CorrEsent^{LA}$ and $CorrTerm^{LA}$ are supposed to be defined as above, in 212.1.
Then, as earlier, $CorrEsent^{LA}$ determines a partition of $CorrTerm^{LA}$ into (three) pairwise disjoint equivalence classes of (correct) LA-terms, called "degrees".
In the case LA := $Q^{-}A$, QA the definition of degrees will slightly differ from that given earlier, in 212.1.

222.2. DEFINITION.

Let B be an LA-compatible site and Δ be an LA-admissible context for B.
If

$$B,\Delta \Vdash_{LA} a$$

then <u>the degree of</u> a (<u>relative to</u> B <u>and</u> Δ <u>in</u> LA) - <u>notation</u> $\underline{\underline{dg}}(B,\Delta,a)$ - will be defined by recursion as follows:

(1) If $a \equiv \tau$ then $\underline{\underline{dg}}(B,\Delta,a) = 1$.

(11) If $a \equiv [v:b_1]b_2$, $\Delta' := \Delta[v:b_1]$ and $\underline{\underline{dg}}(B,\Delta',b_2) = 1$ then $\underline{\underline{dg}}(B,\Delta,a) = 1$.

(2) If $\underline{\underline{dg}}(B,\Delta,b) = n, n \in \mathbb{N}$, and $B,\Delta \vdash_{LA} a:b$ then $\underline{\underline{dg}}(B,\Delta,a) = n+1$.

222.3. REMARK.

Analogues of 212.3., 212.5. and 212.6.(1)-(3) hold for $Q^{-}A$ and QA, as well.
Also, the notation introduced in 212.4. will be employed in the present case.
(Note that $\underline{\underline{dg}}(a)$ is defined only for a in $CorrTerm^{LA}$.)

However, we have already from 222.2. the following specific detail on 1-degrees in $Q^{-}A$ and QA.

222.4.LEMMA.

If a is in $\mathrm{CorrTerm}^{LA}$ and $\underline{\underline{dg}}(a) = 1$ then

either

(1) $a \equiv \tau$ (as in the case of PA and CA)

or

(2) $a \equiv [v_1:a_1]\ldots[v_n:a_n]\tau$, for some $n \geqslant 1$, and LA-terms $a_1,\ldots,a_n$.

Proof. Immediate, from 222.2.⊠

222.5.REMARK.

So in $Q^{-}A$ and QA there is no variable, no head-term and no application term (in $\mathrm{CorrTerm}^{Q^{-}A}$, $\mathrm{CorrTerm}^{QA}$) with degree 1.

On the other hand, if some abstraction term (in $Q^{-}A$,QA) is "correct" and has degree 1 then so is its value-part and the latter has also degree 1.

222.6.REMARK.

Analogues of 212.7., 212.8. and 212.9. hold, by similar arguments (for $Q^{-}A$,QA).
So variables in $\mathrm{CorrTerm}^{LA}$ can have only degrees 2 and 3 and they can be always recovered from appropriate LA-admissible contexts.

222.7.NOTATION.

Hereafter, T (possibly with superscripts - primings -) will range on LA-terms in $\mathrm{CorrTerm}^{LA}$ with degree 1.

222.8.LEMMA.

If B is LA-compatible, Δ is LA-admissible for B and

$$B,\Delta \Vdash_{LA} \underline{c}(b_1,\ldots,b_n)$$

then, for some $\underline{c}$ in Fl_n^{LA}, $n \in \mathbb{N}$, one has

either

(1) $$B,\Delta \vdash_{LA} \underline{c}(b_1,\ldots,b_n) : T$$

or

(2) $$B,\Delta \vdash_{LA} \underline{c}(b_1,\ldots,b_n) : a : T'$$

for some LA-term a (in $\mathrm{CorrTerm}^{LA}$).

Proof. By induction on correctness in LA.⊠

222.9.REMARK.

So, as earlier, in 212.10., head-terms in LA := $Q^{-}A$,QA can have only degrees 2 and 3. Note that, in general, 222.8. holds for both $Q^{-}A$ and QA. Still, 212.10. does not hold, in general, for $Q^{-}A$. But due to the presence of (CI) in QA, 212.10. holds for QA, as well.

222.10.REMARK.

Lemmas 212.12. and 212.13. hold for $Q^{-}A$ and QA,too (and the corresponding proofs are as in the case of PA,CA).

So,it should be already obvious that both $Q^{-}A$ and QA are actually extensions of PA.

As for CA (cf. 212.14. through 212.17.) one can prove the following.

222.11.LEMMA.(arg-degrees in LA := $Q^{-}A$,QA).

If a is an application term in CorrTerm^{LA} then arg(a) is also in CorrTerm^{LA} and dg(arg(a)) = 3.

Proof.By induction on correctness in LA.⊠

However,function-parts of (correct) application-terms in $Q^{-}A$ and QA behave differently (i.e.,not as in CA).

222.12.LEMMA.(fun-degrees in LA := $Q^{-}A$,QA).

If a is an application-term in CorrTerm^{LA} then fun(a) is also in CorrTerm^{LA} and,moreover,

$$2 \leqslant \text{dg}(\text{fun}(a)) \leqslant 3.$$

Proof.By induction on correctness in LA.⊠

Inasfar abstraction-terms in CorrTerm^{LA} (LA := $Q^{-}A$,QA) are concerned one has the following CA-like lemma (cf. 212.22. and 212.23.).

222.13.LEMMA.(dom-degrees in LA := $Q^{-}A$,QA).

If a is an abstraction term in CorrTerm^{LA} then dom(a) is in CorrTerm^{LA} and dg(dom(a)) = 2.

Proof.By induction on correctness in LA.⊠

However one can readily see from 222.4. and 222.5. that value-parts of (correct) abstraction-terms in $Q^{-}A$,QA may also get degree 1 (which was not the case in CA).

222.14.LEMMA.(val-degrees in LA := $Q^{-}A$,QA).

If a is an abstraction-term in CorrTerm^{LA} then val(a) is in CorrTerm^{LA} and

$$1 \leqslant \text{dg}(\text{val}(a)) \leqslant 3.$$

Proof.By induction on correctness in LA.⊠

222.15.COMMENT.

Resuming the degree-considerations on LA := CA,Q^-A,QA one has the following:

	variables	τ	head-terms	application-terms		abstraction-terms	
				$\underline{\underline{arg}}$	$\underline{\underline{fun}}$	$\underline{\underline{dom}}$	$\underline{\underline{val}}$
CA:	2,3	1	2,3	3	3	2	2,3
Q^-/QA:	2,3	1	2,3	3	2,3	2	1,2,3

222.16.REMARK.

As earlier for CA,the intended meaning of application- and abstraction-terms in Q^-A and QA is that:

(1) if a is a (correct) application-term in Q^-A,QA then

$$\underline{\underline{dg}}(a) = \underline{\underline{dg}}(\underline{\underline{fun}}(a))$$

and

(2) if a is a (correct) abstraction-term in Q^-A,QA then

$$\underline{\underline{dg}}(a) = \underline{\underline{dg}}(\underline{\underline{val}}(a)).$$

That is: in LA := Q^-A,QA

(1') correct application-terms can have degrees 2 and 3 only

and

(2') correct abstraction-terms can have any degree (1,2 or 3).

This gives the following "structural" characterization of correct application- and abstraction-terms in Q^-A,QA.

222.17.LEMMA.(LA := Q^-A,QA).

Let B be an LA-compatible site and Δ be an LA-admissible context for B.

If

$$B,\Delta \Vdash_{LA} \{a\}b$$

then <u>either</u>

(1) $\underline{\underline{dg}}(\{a\}b) = 3$

and there are LA-terms b_1 and a' such that

(11) $B,\Delta \vdash_{LA} b : b_1 : [v:a']T$ (i.e.,$\underline{\underline{dg}}(b_1) = 2$)

(12) $B,\Delta \vdash_{LA} a : a' : T'$ (i.e.,$\underline{\underline{dg}}(a') = 2$)

and

(13) $B,\Delta \vdash_{LA} \{a\}b : \{a\}b_1 : T[\![a := v]\!]$

or

(2) $\underline{\underline{dg}}(\{a\}b) = 2$

and there are LA-terms b_1', a'' such that

(21) $b_1' \equiv [v:a'']T$ (i.e., $\underline{\underline{dg}}(b_1') = 1$)

(22) $B,\Delta \vdash_{LA} b : b_1'$

(23) $B,\Delta \vdash_{LA} a : a'' : T'$ (i.e., $\underline{\underline{dg}}(a'') = 2$)

(24) $B,\Delta \vdash_{LA} \{a\}b : T[\![a := v]\!]$

Proof. By induction on correctness in LA, using previous Lemmas.⊠

222.18. LEMMA. ($LA := Q^{-}A, QA$).

Let B be an LA-compatible site and Δ be an LA-admissible context for B.
If

$$B,\Delta \Vdash_{LA} [v:a]b$$

then

(1) $\Delta' := \Delta[v:a]$ is LA-admissible for B

and

(2) there is an LA-term a' such that

(21) $B,\Delta' \vdash_{LA} v:a'$

(22) $B,\Delta \models a \ \underline{\underline{conv}}_{LA} \ a'$

(23) $\underline{\underline{dg}}(a) = \underline{\underline{dg}}(a') = 2$

with also

(3) $B,\Delta' \Vdash_{LA} b$ $1 \leq \underline{\underline{dg}}(b) \leq 3$

(4) $\underline{\underline{dg}}(b) = \underline{\underline{dg}}([v:a]b)$.

Moreover, if $\underline{\underline{dg}}(b) > 1$, then, for some LA-terms b', b'',

(51) $B,\Delta' \vdash_{LA} b:b'$ (i.e., $\underline{\underline{dg}}(b') = \underline{\underline{dg}}(b) - 1$)

(52) $B,\Delta \vdash_{LA} [v:a]b : b''$ (i.e., $\underline{\underline{dg}}(b'') = \underline{\underline{dg}}(b) - 1$)

and

(53) $b'' \equiv [v:a]b'$.

Proof. By induction on correctness in LA, using facts proved earlier.⊠

222.19. REMARK.

Lemmas 212.28. and 212.31. hold also for QA, but not for $Q^{-}A$ (in general).

222.20.REMARK.

It is easy to see that the invariance properties under site- and/or context-expansion established earlier for PA and CA can be also established for $Q^{-}A$ and QA.Moreover,the corresponding proofs are completely similar (mostly,by induction on the definition of correctness in $Q^{-}A$ and QA).

So,in the end,an analogue of Theorem 213.9. can be established for LA := $Q^{-}A$,QA.

That is: the addition of "redundant" (B-correct) LA-constructions to some LA-compatible site and the addition of "redundant" assumptions within some LA-admissible context (such as to preserve LA-admissibility) do not alter the corresponding correctness categories $CorrEsent^{LA}$ and $CorrTerm^{LA}$ (for LA := $Q^{-}A$,QA).This is also true for "derived" correctness categories in LA (as,e.g.,C-,R- and Q-sentences).

222.21.REMARK.

Where LA := $Q^{-}A$,QA correctness is preserved under (simultaneous) substitution in the sense of 214.2. (214.3. and 214.4.) above.

This gives - by the same method as in 214. - "correctness for $Q^{-}A$ and QA-categories" and,finally the following.

222.22.THEOREM.(LA := PA,CA,$Q^{-}A$,QA).

Let B be an LA-compatible site and Δ be an LA-admissible context for B.

If

$$B,\Delta \vdash_{LA} a : b$$

then

$$B,\Delta \Vdash_{LA} a,b$$

Proof.If LA := PA,CA the statement is just 214.6.Else use the fact mentioned in 222.21. and the due $Term_{\alpha}^{LA}$-recursion rules (LA := $Q^{-}A$,QA).⊠

3. Global structure of correctness. PA-separation.

In this section we shall study in some detail general properties of the correctness categories in abstract AUT-languages.
The main aspects to be taken into account will concern:

(1) the "reference order" in LA-compatible sites,

(2) the "complexity" of sound LA-constructions (in LA-compatible sites) and

(3) the "economy of means" in obtaining sound LA-constructions.

The underlying analysis is syntactically oriented and combinatorial in nature. It aims at establishing a kind of "separation property" for the "least" abstract AUT-language PA, while currently manipulating "correctness proofs" in CA, QA, $Q^{-}A$, etc. "PA-separation" is interesting because it allows proving a "conservativity result over PA" for the latter languages.

30. What is going on: heuristics.

The significance of the conservation property intended here was already discussed in a larger context, in 04. above. It remains to establish a precise, "technical" acception of "conservativity" as relativized to the (abstract) AUTOMATH-languages and to insure that the latter one would safely transfer - without loss in precision - to the corresponding "reference"-AUT-languages.

The discussion following here will - specifically - concern proper extensions of PA. In this respect, CA, QA, $Q^{-}A$ and possibly other abstract AUT-languages not considered explicitely here "extend" PA in two distinct directions:

(1) inasfar well-formedness is concerned (CA say has "more well-formed terms", "more well-formed E-sentences", etc. than PA)

(2) as regards correctness (e.g., CA has "more correctness rules" than PA), while PA is "contained" say in CA, QA, etc. both w.r.t. well-formedness and w.r.t. correctness; cf. the initial discussion in 22.).

In order to be able to characterise such extensions as being "conservative over PA" we should, first of all, know how to "isolate" PA *qua* language when it is "used" as a (proper) part of CA, QA, etc.

Specifically, if B is an LA-admissible site (where LA is a proper extension of PA in the sense above) we should know, from the language definition, what is for B "to be in (the language) PA". But it may turn out B could have been "constructed" as a site *in* PA, as well (according to the language definition of PA). That is: every construction in B would be already a PA-construction (according - always- to the efinition of PA).Does this entail that B is already PA-admissible?

n more detail: let B be an LA-admissible site (where LA properly extends PA) and be some LA-construction in B.Then a similar question may arise,viz. what is for "to be in PA" or "to be a PA-construction"? A possible answer would be: look at ts "components" (both immediate and remote);if every such a "component" is "in PA" hen so should be k itself.Clearly,if some "component" of k is either an application erm or an abstraction term,k is "not in PA".Suppose this is not the case;what about he floating constants occurring as "remote components" of k? The LA-construction k ay well "contain" no application and abstraction terms and still be only "apparent-y in PA".Indeed,B was supposed to be LA-admissible.Now if some floating constant c ccurs as a "remote component" of k (k in B) one should have ("by construction") hat c ≡ idf(k') for some other LA-construction k' in B.So the question about the 'PA-ness" of k is reduced to that of whether some other LA-construction k' is "in A" or not.

ne should readily guess that some recursion is needed in order to "isolate" PA ua language whenever it is "used" incidentally in a "larger context".

he exact form of induction necessary here will be explained in 314. below.

oughly speaking,in order to "locate" some LA-construction (where LA is an exten-ion of PA) "on PA" (= the technical meaning of "being in the language PA") one as to "trace back" or to "restore" the "definitional history" of k.
he main point in 314. is that no delta-reduction or reduction to PA-normal-forms re actually necessary along this process.Rather,the underlying analysis can be one in some "rigid" way (combinatorial in nature),relying exclusively on the yntactic form of LA-constructions (that are known to be elements of LA-compatible ites).

ow,where "being on PA" has the technical meaning introduced in 314.,one can define onservativity (for extensions of PA) as follows:

onservativity over PA.

Let LA be any (rule-)extension of PA.Then LA is a conservative extension of PA if,for any LA-compatible site B and any k in B such that k is an LA-construction on PA,there is a PA-compatible site B(k) such that k is in B(k).

It will turn out (33. below) that CA,QA and $Q\bar{}A$ are conservative extensions of PA in the acception above,and,moreover,that the existential claim involved there may be taken in some constructive sense.

In fact, what is actually proved in 33. is a property which is somewhat stronger than conservativity over PA; viz. a PA-separation property.

This can be readily phrased as follows (for LA := CA,QA,Q^{-}A,etc.,as above):

PA-separation property.

Let B be any LA-compatible site and k be an LA-construction in B.

If k is an LA-construction on PA then there is a subset B_k^v of B such that

(1) B_k^v is PA-compatible,(and therefore a site on PA)

(2) k is in B_k^v

and,moreover,

(3) B_k^v can be obtained effectively from B and k and

(4) it is the least subset of B (i.e.,both minimal and uniquely determined) satisfying (1),(2) and (3) above.

It is obvious that the latter property implies conservativity over PA,in the intended sense.

The main part of this section is devoted to the proof of PA-separation.

As to the methods used they may be also relevant in some other respects. Specifically,our main concern consisted - first - of isolating the "real ordering" of LA-constructions in LA-compatible sites (where LA is any abstract AUT-language - as taken into account here).This is (what we called) the reference order in LA(-compatible)-sites and amounts to a precise "coding" of the "definitional history" of LA-constructions (occurring as sound constructions in LA-compatible sites). The reference order in LA(-compatible)-sites is evidentiated by some "tree-like" analysis of (LA-)correctness (cf. 31. below).

It turns out that each LA-compatible site can be viewed as a poset with respect to its reference order (321.).

Relying on this order,a precise (inductive) "measure of complexity" for sound LA-constructions is introduced (322.; the "depth of a sound LA-construction in some LA-compatible site") and - thereafter - used in proofs on LA-correctness.

The abstract approach taken here seems to be significant in some other direction (more or less related to a possible model-theoretic investigation of the AUT-family of languages):it is relatively easy to establish that,for any LA-compatible site B, the family of its LA-compatible subsets is a T_0-topology on B (323.).

Finally,in connection with what has been said in 02.(c_1),some attention has been given to the "economy of means" in obtaining sound LA-constructions.

The latter kind of problem did already occur in work on "reference"-AUTOMATH;namely

in connection with the task of "excerpting" a given correct (AUT-QE-,say) book with respect to a given line k it contains (as a correct line with respect to B), such as to eliminate the "redundant" details (from B),not necessary for the "correctness of k with respect to B" (see,e.g.,JUTTING 79-46,*3.3.Excerpting*., DE BRUIJN 80-72,5.*Processing*. and our discussion in 321.3.).
In particular,"excerpts of correct books" - corresponding,in an abstract setting, to our *analytic sites*;cf. *32ℓ*. below) are shown to be,again,correct books,confirming an earlier conjecture of N.G.de Bruijn *et al*.

31.Tree-analysis.

In order to evidentiate the combinatorial nature of an (abstract) AUTOMATH language we introduce several kinds of labelled trees supposed to "represent" (well-formed resp. correct) syntactic units of the language.
These trees will be used - after appropriate simplifications - in the analysis of correctness (in abstract languages LA).

311.Well-formedness part.

First we give a straightforward tree-analysis of well-formedness in LA's.
The analysis is "global" in the sense that peculiarities in the behaviour of LA's won't actually appear at this level (save maybe those induced by the choice of the primitive alphabet).

311.1.DEFINITION.

(1) The set of well-formed expressions (wf expressions, for short, or even wfe's) of/in LA is a subset Wf^{LA} of Word(LA), viz. the union of the following sets (we suppress here the superscripts "LA", for convenience):

Fl, $\{\square\}$,
Term,
Esent,
$Svar_n$, $Sterm_n$, $Contx_n$, $n \in \mathbb{N}$,
Constr.

(2) Any element of of Wf^{LA} is a wf expression of/in LA.

(3) The variables, the universe symbol, the nil symbol and the floating constants of LA are (all and the only) atomic wfe's of/in LA; the remaining wfe's of/in LA are non-atomic.

311.2.REMARK.

Note that the constructors of LA are not wfe's in LA and they are, clearly, the only structured symbols in the alphabet of LA to share this feature.

311.3.COMMENT.

If X is a wfe in LA let us write leftmost(X) for the structured symbol occurring leftmost in X and call it the free head of X.
We have then, for any X in Wf^{LA}, that

$$\underline{\underline{leftmost}}(X) = \begin{cases} X, \text{ if } X \text{ is atomic} \\ \text{a constructor in } A_{LA} \text{ (the alphabet of LA), else,} \end{cases}$$

so leftmost is a (total) epifunction.
As presented here the (free and canonical) syntagmatic grammar of LA has the following (intended) "functional aspect".
For any generated (free or canonical) syntagmatic category of LA, $Synt^{LA}$ say, if X is a non-atomic wfe in $Synt^{LA}$ then leftmost(X) is a constructor (in the alphabet of LA) and, clearly, the syntactic category of X ($Synt^{LA}$ here) can be retrieved by a mere inspection of its free head symbol. (Of course, this happens only if we know X is a wfe in LA, otherwise leftmost may give either misleading or irrelevant information as an output.)
The reason we had to prefer the "functional" ("direct Polish", or "Łukasiewicz parentheses-free") parsing above (cf. DE BRUIJN 78-61; especially: "Lezen en schrijven van formules") over any other kind of infix-cum-parentheses parsing/notation is in that the former one translates well and somewhat directly into the "language" of labelled trees.
This feature of the syntax of the (abstract AUTOMATH) languages LA will be copiously exploited below.

We define now by "structural" induction, for each wfe in LA, an oriented, labelled tree associated to it (this will be a "planted tree"), viz. the generating tree of a wfe in LA. With a few notable exceptions, to be justified later, the definition follows the (canonical) parsing for wfe's, largely discussed earlier and illustrates the rôle of the epifunctions introduced so far.

311.4. NOTATION.

If X is in Wf^{LA} then $\underline{\underline{W}}(X)$ stands for the generating tree of X.

311.5. DEFINITION.

(0) If X is either
 a variable or
 the universe symbol or
 the nil symbol or
 a floating constant in LA,
then

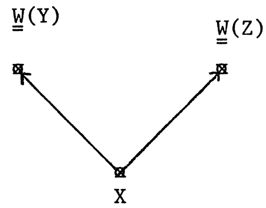
$$\underline{\underline{W}}(X) := \underset{X}{\boxtimes}$$

and the bottom-label of $\underline{\underline{W}}(X)$ is X.

(1) If X is in $Term^{LA}$ and

(11) X is a head-term in LA then

W(X) :=



where

Y := head(X) and either

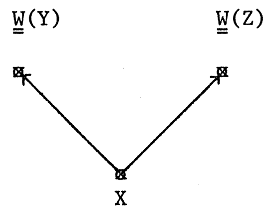
Z := vtail(X), if X is in $Cterm^{LA}$ or

Z := tail(X), else,

and the bottom-label of W(X) is X;

(12) X is an application-term in LA then

W(X) :=



where

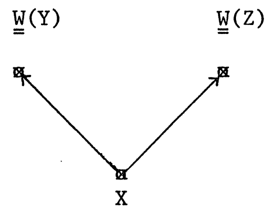
Y := fun(X),

Z := arg(X)

and the bottom-label of W(X) is X;

(13) X is an abstraction-term in LA then

W(X) :=



where

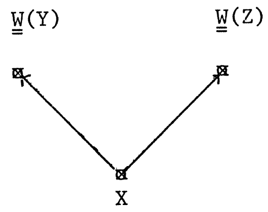
Y := abs(X),

Z := val(X)

and the bottom-label of W(X) is X.

(2) If X is in $Esent^{LA}$ then

$\underline{\underline{W}}(X)$:=



where

Y := $\underline{\underline{sub}}(X)$,

Z := $\underline{\underline{pred}}(X)$

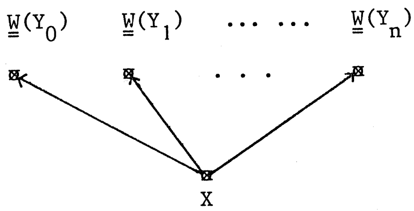
and the bottom-label of $\underline{\underline{W}}(X)$ is X.

(3_n) If X is in $Svar_n$ resp. ($n \in \mathbb{N}$)

(4_n) if X is in $Sterm_n^{LA}$ ($n \in \mathbb{N}$)

then

$\underline{\underline{W}}(X)$:=



where

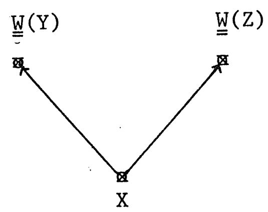
Y_i := $\underline{\underline{elt}}_i^n(X), 0 \leq i \leq n$,

and the bottom-label of $\underline{\underline{W}}(X)$ is X.

(5_n) If X is in $Contx_n^{LA}$ ($n \in \mathbb{N}$)

then

$\underline{\underline{W}}(X)$:=



where

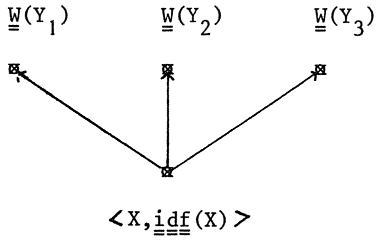
Y := $\underline{\underline{str}}_1(X)$,

Z := $\underline{\underline{str}}_2(X)$

and the bottom-label of $\underline{\underline{W}}(X)$ is X.

(6) If X is in $Constr^{LA}$ then

W(X) :=



where

Y_1 := ctx(X),

Y_2 := def(X),

Y_3 := cat(X)

and the bottom-label of W(X) is the pair ⟨X,idf(X)⟩.

311.6.COMMENT.

It is easy to check that the induction underlying the previous definition "closes" well;so we have a good inductive definition.
A stronger way of "closing" will be introduced next in view of analysing correctness in LA-sites.

311.7.REMARK.

The tree-analysis of wfe's in 311.5. respects the original syntactic parsing of section 1. except for LA-terms (viz. head-terms and abstraction-terms) and LA-constructions.
In the case of head- resp. abstraction-terms a the generating trees W(a) contain essentially the same kind of information as "coded" via the primitive (free/canonical) parsing: just use the clauses for variable- or term-strings and that for E-sentences in order to see this.
The way of analysing constructions above does not "loose" information either, when compared with the original (canonical) parsing ("identifiers" are implicitely present at bottoms).To see the reason behind this choice cf. below the extension to "analytic trees".

312. Analytic trees.

The generating trees of wfe's in LA are best suited to analyse the syntactic units of LA "in isolation", i.e., disregarding their "site-environment". Next we shall relativize the tree-analysis of well-formedness above to the case of correctness. That is: LA-terms, variable- and term-strings in LA, LA-contexts and LA-constructions will be supposed to "occur in" LA-compatible sites.

Specifically, we shall associate to each wfe ("occurring") in a given LA-compatible site a labelled, oriented tree, which will be seen to "codify", in a precise sense, both the syntactic structure of that wfe and (some of) the "definitional history" of its "components".

The latter trees - hereafter called analytic trees (of wfe's) - arise, roughly speaking, by "composition", from appropriate generating trees of wfe's ("occurring in" some LA-compatible site B).

312.1. NOTATION.

As earlier, if X is in Wf^{LA} then $\underline{\underline{W}}(X)$ is the generating tree of X; also if X is as above then $\underline{\underline{T}}(X)$ stands for the analytic tree of X, where it is supposed that X is a wfe "occurring in" some LA-compatible site B. (Here "LA" may, again, stand for any abstract AUTOMATH language considered earlier.)

312.2. DEFINITION.

(1) If X is in $Wf^{LA} - Fl^{LA}$ then

$$\underline{\underline{T}}(X) := \underline{\underline{W}}(X),$$

and bottom-labels are preserved.

(2) If X is a floating constant in LA and Y is a canonical construction in LA ("occurring in" the same LA-compatible site as X does) such that $X \equiv \underline{\underline{idf}}(Y)$, then

$$\underline{\underline{T}}(X) := \underline{\underline{T}}(Y),$$

and the bottom-label of $\underline{\underline{T}}(X)$ is the same as the bottom label of $\underline{\underline{W}}(Y)$ (that is: it is the pair $\langle Y, \underline{\underline{idf}}(Y) \rangle := \langle Y, X \rangle$; cf. 311.5.(6) above).

312.3.COMMENT.

The intention behind the previous definition is to "encode" in an intuitive way (nonetheless precise) those aspects of the "definitional history" of a wfe that depend only on syntactic considerations in LA disregarding superimposed ingredients "forcing" the syntax by a détour via the associated reduction system (LA) or by "collapsing information" on types/categories (i.e.,by applications of the Rule of Category Inclusion in QA-languages). Thus the analytic tree of a wfe X in LA may not always furnish a complete record of the "definitional history" of its "components" (here and everywhere in the sequel,the "components" of a wfe are to be understood - more or less formally - as resulting from a "syntactic" analysis via generating trees).

Specifically,the cases where this analysis fails to be "complete" are exactly those involving:

(1) in any LA: some application of one of the Rules of Category Conversion,

(2) in QA-languages: some application of the Rule of Category Inclusion.

Let,for any language LA introduced earlier,LA^o be the sub-language of LA we get by eliminating the Rules of Category Conversion from the primitive setting.

Then it should be,by now,clear that the analytic trees of wfe's in

$$PA^o, \beta\text{-}CA^o, \beta\eta\text{-}CA^o, \beta\text{-}Q^-A^o, \beta\eta\text{-}Q^-A^o, \text{etc.}$$

do actually provide a complete analysis of the "definitional dependencies" for wfe's "occurring in" sites that are compatible in the restricted languages.

However - as pointed out by N.G.de Bruijn - we hardly get something interesting,from both a theoretical and a practical viewpoint,by manipulating LA^o-languages instead of the corresponding "full" ones.

This motivates the next step in the tree-analysis of correctness.

(Analytic trees are nevertheless very useful since,actually,only such trees will be manipulated in a deeper analysis of the correctness categories of LA's.Cf.32. et sqq.)

It is relatively easy to check that 312.2. is a good inductive definition.To see that it "closes well" one would however need the following Lemma.

312.4.LEMMA.

Let B be an LA-compatible site (LA := PA,CA,$\bar{Q}$A,QA,etc.) and k be a construction in B.Then idf(k) is rank-fresh for the sequence

(i) $\langle a_1,\ldots,a_n,a \rangle$, where a := cat(k),if k is a p-construction

(ii) $\langle a_1,\ldots,a_n,a,b \rangle$, where a := cat(k),b := def(k),if k is a d-construction

and,where,in both cases, lh(ctx(k)) = n and,whenever $n \geqslant 1$,one has

$$a_i := \mathrm{elt}_i^n(\mathrm{str}_2(\mathrm{ctx}(k))) \qquad (1 \leqslant i \leqslant n).$$

Proof.This is the rôle of the restriction on the ranks of idf's in any $\mathrm{Site}^{LA}_{\boxtimes}$-recursion rule.⊠

312.5.REMARK.

Therefore,the closure condition in 312.2. won't give rise to "cycles". Indeed,for any k in some LA-compatible B,idf(k) cannot occur in ctx(k),or in cat(k) or in def(k) - if,in the latter case,k is a d-construction -. Moreover,due to the recursiveness of the correctness categories of LA (in particular,$\mathrm{Site}^{LA}_{\boxtimes}$ is defined inductively),one cannot encounter bigger "cycles".

In terms of (analytic) trees this gives the following consequence.

312.6.COROLLARY.

Let B be an LA-compatible site with k in B.Then for any LA-construction k' (in B) such that $k \not\equiv k'$ and T(k') is a sub-tree of T(k),one has

$$\mathrm{rank}(\mathrm{idf}(k')) < \mathrm{rank}(\mathrm{idf}(k)).$$

Proof.Immediate from 312.4. and the definition of the analytic trees (312.2.) above.⊠

313. Genetic trees: a digression.

As we did not adopt any "algorithmic" point of view in the definition of the languages LA (this is certainly possible, using, mutatis mutandis, the pattern of VAN DAALEN 80-73, V.4. and VII., say) the only way of "deciding" the compatibility of some site in LA we have at hand should consist of displaying, in each particular case under consideration, a (finite) sequence of "applications" of the correctness rules of LA to wfe's in LA. This is, in the end, very cumbersome a book-keeping and a few exercising would quickly convince the reader of the fact the needed lists of rules accompanied by local hints, as regards the particular inputs they should take in each case, are completely unpractical (the necessary book-keepings are much longer than one would expect at a first look).

Somewhat more practical a solution in analysing the correctness of (constructions in) LA-sites would consist of relying on analytic trees - for the main part of the process - in some "derived" sense; viz. by "book-keeping" only particular applications of the Rules of Category Conversion (for all LA's) and, in the case of QA-languages, the applications of the Rule of Category Inclusion. Obviously, if both kinds of information (viz. that present in an analytic tree and that present in the - more economical - book-keeping of the applications of the rules indicated above) can be conveniently "stored", we would be eventually able to restore the "definitional history" of the LA-constructions concerned and to supply a "correctness proof" for them (within the appropriate environment).

The main question is whether "completing" an analytic tree by the missing information (cf. 312.3.) would possibly amount to some manageable setting for analysis, comparable in economy with that encountered in the case of analytic trees say.

The answer is, fortunately, affirmative: it will be seen that the "completion" mentioned above can be done such as to preserve the tree-like search ("in(to) the definitional history" of a wfe in LA).
Still, such a "completion" may not be always uniquely determined. Indeed, we might obtain the same LA-compatible site by taking into account different sequences of applications of the correctness rules of LA to wfe's in LA, such that the "book-keeping" of the applications of the rules (CC_i), $i = 1,2$ and (CI) - in the case of QE - would lead to possibly distinct "correctness proofs".

Let first illustrate what kind of "incompletensess" of the analytic trees we have in mind.

Suppose B is an LA-compatible site (the choice of LA is immaterial) and k is a (canonical) construction in B.
One of the "ascending branches" starting from the bottom of $\underline{\underline{T}}(k)$ will always "point out" to $\underline{\underline{cat}}(k)$, which is - always - an LA-term.
Now it may well happen that this LA-term (a := $\underline{\underline{cat}}(k)$, say) has been obtained, by an application of one of the Rules of Category Conversion, from <u>some</u> LA-term a', whence we must have, <u>either</u>

(1) a' $\underline{\underline{contr}}_{LA}$ a <u>or</u>

(2) a' $\underline{\underline{exp}}_{LA}$ a (that is: a $\underline{\underline{contr}}_{LA}$ a'),

and $\underline{\underline{contr}}_{LA}$ (resp. $\underline{\underline{exp}}_{LA}$) may be either $\underline{\underline{contr}}_{\delta}$, or $\underline{\underline{contr}}_{\beta}$ or $\underline{\underline{contr}}_{\eta}$ (and analogously for $\underline{\underline{exp}}_{LA}$).

If the <u>tree-analysis</u> of k above is to reflect accurately <u>the way</u> k <u>was obtained as a sound construction in</u> B (and not "checked", say, <u>via</u> the "algorithmic definition" or something similar to the program described in ZANDLEVEN 73-36) it is not just a (:= $\underline{\underline{cat}}(k)$, above) we have to analyse further, but rather the LA-term a', from which we previously got a, by an application of Category Conversion.

Tu put things otherwise: in order to obtain a "complete record of definitional history" inasfar k is concerned, we should not go on analysing $\underline{\underline{T}}(a)$, with a := $\underline{\underline{cat}}(k)$, but rather $\underline{\underline{T}}(a')$, repeating the "jumps" with any <u>previous</u> application of a Rule of Category Conversion. This will, in the end, insure <u>a recursive search on the structure of</u> k whenever (CI) is not present (e.g., <u>either</u> if LA is not a QA-language <u>or</u>, in β-QA, $\beta\eta$-QA, etc., whenever the Rule of Category Inclusion was not applied in obtaining k).

To make the description of the resulting trees complete we have to repeat the process above for particular applications of (CI) - in QA-languages -, except that, in the latter case, we must "jump" with any application of (CI) "on the way of obtaining" k as a sound construction in B.

All this is just to give a feeling about the motivations behind the way we are going to "complete" (or if one prefers, to "complicate") the analytic trees of wfe's in the sequel. Strictly speaking, the heuristic remarks above are <u>theoretically dispensable</u>.

Next we define for each wfe "occurring in" some LA-compatible site B a labelled, oriented tree, looking very much like the analytic tree of that wfe. The corresponding trees will be called genetic trees of wfe's.

We introduce first some notation and terminology.

313.1. CONVENTION. NOTATION.

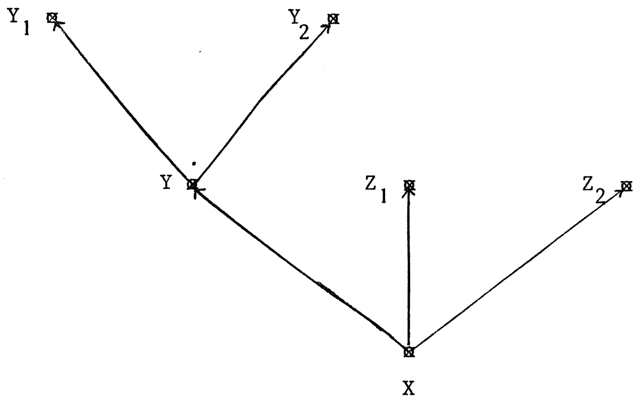
Henceforth, if X is any wfe in LA, any node of $\underline{\underline{W}}(X)$, $\underline{\underline{T}}(X)$ resp. will be confused with the label associated to that node.

It is also supposed to be known what are, for a given node Y in some $\underline{\underline{T}}(X)$ say, with X a wfe in LA,

(1) the (set of) immediate successors of Y; notation: $\underline{\underline{isc}}(Y)$, and

(2) the (set of) successors of Y; notation: $\underline{\underline{sc}}(Y)$.

EXAMPLE: Let $\underline{\underline{T}}(X)$ be the following tree (it is not important to discover what it stands for).



Then $\underline{\underline{isc}}(X) = \{Y, Z_1, Z_2\}$, $\underline{\underline{isc}}(Y) = \{Y_1, Y_2\}$, while $\underline{\underline{sc}}(X)$ is the union of $\underline{\underline{isc}}(X)$ and $\underline{\underline{isc}}(Y)$.

313.2. TERMINOLOGY.

In order to refer conveniently to applications of the Rules of Category Conversion and Category Inclusion we introduce the following way of speaking.

Recall that (CC_1) and (CC_2) were phrased as follows:

If B is LA-compatible and Δ is LA-admissible for B and

$B,\Delta \vdash_{LA} a:b_1$

$B,\Delta \Vdash_{LA} b_1,b_2$

then also

$B,\Delta \vdash_{LA} a:b_2$

provided (1) $b_1 \underline{\underline{contr}}_{LA}\, b_2$ (in (CC_1)); (2) $b_2 \underline{\underline{contr}}_{LA}\, b_1$ (in (CC_2)).

For any application of (CC_i), $i = 1,2$ we say that

Φ_1 := $a:b_1$ is the input of that application of (CC_i) and

Φ_2 := $a:b_2$ is the output of the application concerned,

with analogously,

b_1 as category-input and

b_2 as category-output of the corresponding application of (CC_i).

Similarly, the Rule of Category Inclusion (for QA-languages), i.e., (CI), was:

If B is QA-compatible and Δ is QA-admissible for B and

$B,\Delta \vdash_{QA} b:T_1$

then

$B,\Delta \vdash_{QA} b:T_2$

where, as expected,

$T_1 := [v_1':a_1']\ldots[v_m':a_m'][v':a]\tau$; $T_2 := [v_1':a_1]\ldots[v_m':a_m']\tau$.

In this case, we have as earlier, for any application of (CI),

Φ_1 := $b:T_1$ is the input and T_1 is the category-input of the application,

Φ_2 := $b:T_2$ is the output and T_2 is the category-output of the corresponding application of (CI).

313.3. NOTATION. CONVENTION.

If X is a wfe "occurring in" some LA-compatible site B then $\underline{\underline{G}}(X)$ will denote the genetic tree of X (to be introduced next). Moreover, B is supposed to be "given" together with the way it was obtained qua LA-compatible site.

Now the definition of the genetic tree of a wfe X in LA can be completed by:

(1) stating what is $\underline{\underline{G}}(X)$ in case no application of the Rules (CC_i), $i = 1,2$ were involved in the "process of construction" of X (according to the correctness rules of LA) and

(2) by making explicit the due rules of "composition" for generating trees of wfe's in case some LA-term was "modified" by an application of one of the indicated rules along the "process of construction" of X.

We omit below the specification "according to the correctness rules of LA".

313.4. DEFINITION.

(1) If no application of the correctness rules (CC_i), $i = 1,2$ or - in QA-languages - (CI), was involved in the "process of construction" of a wfe X in LA then

$$\underline{\underline{G}}(X) := \underline{\underline{T}}(X).$$

(2) For any application of (CC_i), $i = 1,2$ involved in the "process of construction" of a wfe X in LA, such that its category-input is an LA-term a and

its category-output is an LA-term b,with b in isc(X),the genetic tree G(X) of X is to be obtained from its analytic tree T(X),by replacing the subtree T(b) in T(X) by G(a) and "marking" the replacement by changing the bottom-label of G(a) into a pair ⟨⟨a,b⟩,0⟩.

(3) For any application of (CI) - in QA-languages -,involved in the "process of construction" of a wfe X (in QA),such that its category-input is a QA-term a and its category-output is a QA-term b,with b in isc(X),the genetic tree G(X) of X is to be obtained from its analytic tree T(X) by replacing the subtree T(a) in T(X) by G(a) and "marking" the replacement by changing the bottom-label of G(a) into a pair ⟨⟨a,b⟩,1⟩.

313.5.COMMENT.

The "tree-replacement" operations described in 313.4. above produce,obviously, a recursive definition of G(X),for any wfe X "occurring in" some LA-compatible site.

The order in which we have to perform these operations is,however,not immaterial and one can easily see that the choice of this order is severely restricted by the very formulation of the "rules of tree-replacement" stated in Definition 313.4. (There is still some "choice" - as we shall indicate below - but the "alternatives" are rather irrelevant and we can make easily the process of "tree-replacement" completely "deterministic".Cf. 313.7.

To make this very explicit we need some more terminology (local,this time).

313.6.TERMINOLOGY.

Let T be a (finite) "planted tree" (so T is an oriented tree with a "root"; in the above the "root" was called "bottom-node").

The paths in T (or the T-paths) are supposed to "connect" a "top-node" (= a "leaf") in T to/with its bottom;they will be viewed as sequences of nodes in T.

Let a be a node in T.A T-path of a ia a T-path containing a (viz.,a "lays on" any one of its T-paths) and will be (locally) denoted by p(T,a).

Every node in T is on a finite number,$n \geq 1$,of T-paths $p_j(T,a), 1 \leq j \leq n$.

Let $lh(p_j(T,a))$ be the length of the T-path $p_j(T,a)$;this is just the number of nodes "laying on" the path,including the "bottom-node".

Define,for any node a in T,

$$lh(T,a) = max\{lh(p_j(T,a)): \ 1 \leq j \leq n\}$$

where $p_j(T,a), 1 \leq j \leq n$, are all and the only T-paths of a.

The number of nodes in $\underline{\underline{T}}$ "connecting" $\underline{\underline{a}}$ to/with the bottom of $\underline{\underline{T}}$ and distinct from $\underline{\underline{a}}$ itself is the distance of $\underline{\underline{a}}$ (to the bottom) in $\underline{\underline{T}}$;notation: $\underline{\underline{dist}}(\underline{\underline{T}},\underline{\underline{a}})$. Clearly,$\underline{\underline{dist}}(\underline{\underline{T}},\underline{\underline{a}})$ does not depend on some particular $\underline{\underline{T}}$-path of $\underline{\underline{a}}$,since it "measures" only the sub-sequence the $\underline{\underline{T}}$-paths of $\underline{\underline{a}}$ "share" in common.
Now the $\underline{\underline{T}}$-depth of a node $\underline{\underline{a}}$ is

$$\underline{\underline{depth}}(\underline{\underline{T}},\underline{\underline{a}}) = \underline{\underline{lh}}(\underline{\underline{T}},\underline{\underline{a}}) - \underline{\underline{dist}}(\underline{\underline{T}},\underline{\underline{a}}).$$

So,"top-nodes" (= "leaves") in a finite planted tree $\underline{\underline{T}}$ will have always $\underline{\underline{T}}$-depth 0,they are immediate successors (in $\underline{\underline{T}}$) of nodes with $\underline{\underline{T}}$-depth 1 (provided the latter exist) and so on,until reaching the bottom of $\underline{\underline{T}}$.
Clearly,the $\underline{\underline{T}}$-depth of any node $\underline{\underline{a}}$ (of $\underline{\underline{T}}$) is uniquely determined (in $\underline{\underline{T}}$).
Finally,let a $\underline{\underline{T}}$-layer be the set of nodes in $\underline{\underline{T}}$ having the same $\underline{\underline{T}}$-depth and call this $\underline{\underline{T}}$-depth the $\underline{\underline{T}}$-depth of the ($\underline{\underline{T}}$-)layer.

313.7.COMMENT.

Using the terminology of 313.6. we can easily establish that,for any wfe X "occurring in" some LA-compatible site,the genetic tree of X has to be constructed "by induction on $\underline{\underline{T}}$-depths",where $\underline{\underline{T}}$ is here actually $\underline{\underline{T}}$(X),i.e.,the analytic tree of X.This can be done by fixing an arbitrary order on each $\underline{\underline{T}}$(X)-layer (from-left-to-right say,using the orientation of the tree in the plane; note that this is the only freedom we have when performing the "tree replacements" stipulated by 313.4.).
Explicitely,one has to start at the top-nodes of $\underline{\underline{T}}$(X),i.e.,with the $\underline{\underline{T}}$(X)-layer of depth 0,exhaust the layer by applying clauses (1)-(3) of 313.4.,then increase depths by one and go on applying theses clauses ("rules") until the maximal $\underline{\underline{T}}$(X)-depth is reached (i.e.,at the bottom of $\underline{\underline{T}}$(X),which - in fact - is/bears the label X).
It is easy to see that the underlying "algorithm" can be always applied to any analytic tree $\underline{\underline{T}}$(X),provided a "book-keeping" of the applications of the rules (CC_i),i = 1,2 and (CI) is given (for the construction of the site).
Moreover,the resulting genetic tree(s) will contain both the information present in the original (syntactic) analytic tree and that contained in the appropriate "book-keeping" of the applications of (CC_i) and (CI),if any.

313.8.REMARK.

In the definition of genetic trees we have adopted - deliberately - an "intensional" point of view on LA-compatibility (in sites).
Specifically,it was supposed that, whenever some LA-site B is known to be LA-compatible, we also have at hand some record of the way it was obtained.
In other words,any LA-compatible site B was supposed to be "given" together with some modus operandi for establishing its LA-compatibility via some finite sequence of applications of the correctness rules of LA.
Now it might well happen that for one and the same LA-compatible site B we had at hand two distinct sequences of applications of the correctness rules of LA.If theses "differences" would incidentally concern the number of applications of (CC_i),i = 1,2 and/or (CI) or even the "places" where these rules were applied (in the analytic tree to start with) one should certainly obtain genetic trees - for the same wfe "occurring in" B - which would differ in structure.
So the genetic tree of a wfe reflects accurately - maybe even too accurately - the "definitional history" of that wfe,in the sense it takes into account details in the "process of its construction" (according to the correctness rules of LA) that could have been completely different nonetheless reaching ("extensionally") the same (LA-compatible) site.
It is therefore correct to speak about "a genetic tree" of a given wfe X "occurring in" some LA-compatible site and not about "the genetic tree" of the wfe X.
Note that the same kind of "intensional analysis of correctness" is implicitely present in proofs "by induction on the definition of correctness in LA".In the latter case the underlying induction is done "on the length" of some list "book-keeping" the applications of the correctness rules of LA - no matter which one and disregarding possibly different lists which have,so to speak,the same "effect".

314.On isolating wfe's "on" PA.

From the language definition we should know,in general,for any syntagmatic category $Synt^{LA}$ of LA,what is for a wfe in that category to be in LA by,and only by, "structural induction".In particular,drawing genetic trees of wfe's would eventually lead to a non-ambiguous identification of the language which a given wfe belongs to.This information is essentially obtained by "positive induction": if LA is not PA say we cannot "regress inductively" in order to get information about the wfe's that "are in" the set-theoretic difference LA - PA (i.e.,we cannot identify by a mere inductive argument those wfe's that are not in PA,nonetheless being wfe's in LA).This is also true for LA-sites (where LA is not PA).

Of course we may identify "empirically" wfe's and sites in LA (if LA is not PA) as being on PA,in each particular case and as,in each such a case,the information we have to "process" is finite,we can finally recover the things that are not on PA, by some appropriate book-keeping performed simultaneously with the tree-search for things that are on PA.Still,that is not what we may want and actually need in proving facts about LA versus PA.Specifically (if,again,LA is not PA) we may need an analogous "structural" way of searching for wfe's (and sites) that are,ultimately,in LA without already being in ("on") PA (i.e.,for things that are properly in LA $\neq$ PA).

A possible - and reasonable - way out (and we couldn't find a simpler alternative to what follows) is to specify,by positive induction,for any wfe (resp. site) in LA both the meaning of being-on-PA and the meaning of not-being-on-PA (for any LA;since "not-being-on-PA" is empty for wfe's/sites in PA).
So we can obtain inductively (where the induction is as intended,viz. "structural") disjoint sets of wfe's PA^+ and PA^- say,enabling us to rely on a "regressive search" throughout an LA-site in order to establish the "PA-ness" or the "non-PA-ness" of a given wfe occurring in that site.[1)]

Resuming: we need a precise inductive definition of what is to be meant by
(1) "being on PA" and (2) "not being on PA",
for LA-sites,and,in general,for wfe's in LA.To this effect we define disjoint

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1) We have learnt the advantages of the approach from Peter Aczel,but the idea is familiar in literature,being somewhat more than mere "folklore" on inductive definability.Dana Scott has employed similar methods in order to introduce the truth-predicates for the three-valued system of logic presented in 1975,at the Rome Lambda-conference (see SCOTT 75a).

classes of wfe's in LA, corresponding resp. to (1) and (2) above.
This will be achieved, as announced, by positive induction, using implicitely the tree-analysis of well-formedness/correctness discussed at length above.

314.1.NOTATION.COMMENT.

Let X be a wfe in LA (where one can agree, for convenience, that LA is not PA), "occurring in" some LA-compatible site B.
We define for each such a B, subsets $\underline{PA}^+$ and $\underline{PA}^-$ of Wf^{LA}.
Interpretation:
"X is in $\underline{PA}^+$" is a formal explicatum of "X is a wfe on PA" and
"X is in $\underline{PA}^-$" is a formal explicatum of "X is a wfe on LA - PA" (i.e., X is in $\underline{PA}^-$ iff X is a wfe in LA and X is not a wfe on PA).

314.2.DEFINITION.

(01) If X is either a variable or the universe symbol or the nil symbol then X is in $\underline{PA}^+$.

(02) If X is a floating constant and X ≡ idf(Y), for some sound construction Y in B then
X is in $\underline{PA}^{\pm}$ iff Y is in $\underline{PA}^{\pm}$.

(1) If X is in $Term^{LA}$ and, specifically,

(111) X is a canonical head-term then
X is in $\underline{PA}^+$ iff head(X) is in $\underline{PA}^+$,
X is in $\underline{PA}^-$ iff head(X) is in $\underline{PA}^-$

(112) X is a non-canonical head-term then
X is in $\underline{PA}^+$ iff both head(X) and tail(X) are in $\underline{PA}^+$,
X is in $\underline{PA}^-$ iff either head(X) or tail(X) or both are in $\underline{PA}^-$;

(12) X is an application-term then
X is in $\underline{PA}^-$;

(13) X is an abstraction-term then
X is in $\underline{PA}^-$;

(2) If X is in $Esent^{LA}$ then
X is in $\underline{PA}^+$ iff both sub(X) and pred(X) are in $\underline{PA}^+$,
X is in $\underline{PA}^-$ iff either sub(X) or pred(X) or both are in $\underline{PA}^-$.

(3_n) If X is in $Svar_n$ ($n \in \mathbb{N}$) then
X is in $\underline{PA}^+$.

(4_n) If X is in $Sterm_n^{LA}$ ($n \in \mathbb{N}$) then
X is in $\underline{PA}^+$ iff, for all $i, 0 \leq i \leq n$, $elt_i^n(X)$ is in $\underline{PA}^+$,
X is in $\underline{PA}^-$ iff, for some $i, 1 \leq i \leq n$, $elt_i^n(X)$ is in $\underline{PA}^-$.

(5_n) If X is in $Contx_n^{LA}$ ($n \in \mathbb{N}$) then

X is in $\underline{PA}^+$ iff both $\underline{str}_1(X)$ and $\underline{str}_2(X)$ are in $\underline{PA}^+$,

X is in $\underline{PA}^-$ iff either $\underline{str}_1(X)$ or $\underline{str}_2(X)$ or both are in $\underline{PA}^-$.

(6) If X is in $Constr^{LA}$ then

X is in $\underline{PA}^+$ iff $\underline{ctx}(X)$,$\underline{cat}(X)$ and $\underline{def}(X)$ are in $\underline{PA}^+$,

X is in $\underline{PA}^-$ iff at least one of $\underline{ctx}(X)$,$\underline{cat}(X)$,$\underline{def}(X)$ are in PA^-.

Finally,we have the following.

314.3.DEFINITION.

An LA-compatible site B is said to be

(1) a site on PA if every construction k in B is in $\underline{PA}^+$,

(2) a site on LA if there is some construction k in B such that k is in $\underline{PA}^-$.

314.4.REMARK.

If two LA-terms a,b differ only by some alpha-conversions and a is in $\underline{PA}^{\pm}$ then we will also let b be in $\underline{PA}^{\pm}$.

314.5.TERMINOLOGY.

The "on"-terminology will be used in connection with any wfe in LA (especially for constructions).E.g.,we say that k (k in some LA-compatible site B) is a construction on PA (resp. on LA,whenever LA is not PA) iff k is in $\underline{PA}^+$ (resp. in $\underline{PA}^-$).

314.6.REMARK.

There is no point in distinguishing here between β-LA and $\beta\eta$-LA.For the moment being we shall agree that the way of speaking introduced previously applies to both "extensionality types".

314.7.COMMENT.

It is easy to see that a proper use of "on" above presupposes that the syntactic unit concerned is already "correct" or "occurs in" some LA-compatible site. (Otherwise the underlying induction won't work.)

Of course,for syntactic units not involving floating constants "local identifications" of the language they "belong" to would be possible without paying attention to matters of correctness.(There are not too many things left within the latter rubric,however.)

32. Global structure of (LA-compatible) sites.

Relying on the tree-analysis described above, we shall give a global, abstract characterization of the structure of correctness in LA-sites.

321. Reduced sites. Reference order in LA-sites.

We study first LA-sites that are "minimal" for given (sound) LA-constructions.

321.1. DEFINITION.

Let B be an LA-compatible site and k be a construction in B (LA as earlier).

(1) A reduct of B for k is a minimal subset B_k^v of B such that

(i) k is in B_k^v and

(ii) B_k^v is LA-compatible.

(2) Let B_k^v be as above. Then we say that B is a reduced site for k if

$$B = B_k^v.$$

(3) A reduced LA-site is an LA-compatible site B such that there is some k in B and B is a reduced LA-site for k.

321.2. REMARKS.

We may stipulate, by convention, that the empty site is always a reduced LA-site. For the moment being we do not care about the unicity of reducts/reduced sites. A reduced site B for some k (k should be in B) is its own reduct for k. Reduced sites are, so-to-speak, "minimal" for a given construction they contain. If B is a reduced site for some k than we cannot "reduce" it any further without thereby "loosing" k (i.e., its soundness for B). Here "reduction" has not the technical meaning involved in the notions of reduction discussed earlier. Note also that reduced sites are always (LA-)compatible (and this respects the convention above on the empty site).

321.3. COMMENT.

The abstract notion of a reduced site has a "reference"-AUT counterpart in what L.S. van Benthem Jutting et al. used to call "excerpted text/book" (cf., e.g., JUTTING 79-46, 3.3. Excerpting. and also DE BRUIJN 80-72, 5. Processing.). This has been implemented for AUT-QE in the following sense:

Let $\underset{\wedge}{B}$ be a correct AUT-QE book and call any sub-sequence of $\underset{\wedge}{B}$ a sub-book of $\underset{\wedge}{B}$. A program called excerpt has been devised which, given any correct AUT-QE book $\underset{\wedge}{B}$ and a line ℓ in $\underset{\wedge}{B}$, produces the minimal correct sub-book of

$\underset{\wedge}{\underline{B}}$ containing $\mathfrak{L}$ (that is: it produces the "excerpted book,from $\underset{\wedge}{\underline{B}}$,for $\mathfrak{L}$").
For an example see the excerpted book for Satz 27 (in Landau's Grundlagen der Analysis,as translated by L.S.van Benthem Jutting in AUT-QE),given in JUTTING 79-46,Appendix 4.
It is easily seen that "excerpting" makes sense for (any) other "reference"-AUT language,as well.

Sometimes we shall need the following refinements of the above.

321.4.DEFINITION.
Let B be an LA-compatible site and k be a construction in B such that k is a construction on PA.We then say that B is PA-reducible for k if some reduct B^{v}_{k} of B for k is a site on PA.
If B is PA-reducible for some k we say simply that B is PA-reducible, while a reduct of B (for k) will be specified as a PA-reduct of B (for k) if it is a site on PA.

321.5.REMARK.
Any compatible PA-site is PA-reducible,of course.

We characterise the structure of LA-compatible sites via reduced sites.This will be done by essential use of the tree-analysis discussed earlier.

321.6.DEFINITION.
An analytic/genetic tree $\underline{\underline{H}}$ is a construction-tree if it is an analytic /genetic tree of some LA-construction (that is: its bottom-label is a pair $\langle k,\underline{c}\rangle$, where k is an LA-construction and $\underline{c} \equiv \underline{\underline{\underline{idf}}}(k)$;cf. 311.5.(6) above).

Given an LA-compatible site B,we define relations $\multimap\!\!<$ and $=\!\!<$ on B via construction-trees as follows.

321.7.DEFINITION.
Let B be an LA-compatible site with k_1,k_2 in B and $k_1 \not\equiv k_2$.Then

$$k_1 \multimap\!\!< k_2$$

if

(1) $\underline{\underline{T}}(k_1)$ is a sub-tree of $\underline{\underline{T}}(k_2)$,
(2) for no k in B,distinct from both k_1 and k_2,one can have simultaneously
  (i) $\underline{\underline{T}}(k)$ is a sub-tree of $\underline{\underline{T}}(k_2)$ and
  (ii) $\underline{\underline{T}}(k_1)$ is a sub-tree of $\underline{\underline{T}}(k)$.

321.8.TERMINOLOGY.

The relation $\multimap\!\!<$ will be called the atomic reference order (relation) in B. For k_1, k_2 in some LA-compatible site B, $k_1 \multimap\!\!< k_2$ reads: "k_2 depends directly on k_1", or "k_2 is an immediate successor of k_1 (relative to the reference order in B)" or "k_1 is an immediate predecessor of k_2 (relative to the reference order in B)" or, even, "k_2 covers k_1".

321.9.DEFINITION.

The reference order (relation) in any LA-compatible site B is the reflexive and transitive closure of $\multimap\!\!<$.

Notation: $\preccurlyeq$.

321.10.TERMINOLOGY.

For k_1, k_2 in any LA-compatible site B, $k_1 \preccurlyeq k_2$ reads "k_2 depends on k_1" or k_2 (k_1) is a successor (predecessor) of k_1 (k_2) relative to the reference order in B".

321.11.NOTATION.TERMINOLOGY.

For convenience, we shall also use the irreflexive variant of $\preccurlyeq$, hereafter denoted by "$\prec$".

So for k_1, k_2 in any LA-compatible site B, $k_1 \prec k_2$ reads: "k_2 strictly depends on k_1", etc. and $k_1 \prec k_2$ holds iff $k_1 \preccurlyeq k_2$ holds with $k_1 \not\equiv k_2$ (syntactic inequality, modulo alpha-convertibility whenever necessary, but see the conventions on alpha-conversions in 123. above).

321.12.REMARK.

The reference order in an LA-compatible site describes completely the "dependence"-interrelationship between its elements (LA-constructions). It is obvious that the reference order in any LA-compatible site is antisymmetric.

So we have the following straightforward fact.

321.13.LEMMA.

Any LA-compatible site is a poset relative to its reference order.

Proof. Clear. ∎

321.14.COMMENT.

N.G.de Bruijn used to define,in recent time (in lectures on AUTOMATH),"correct AUTOMATH books" as being posets.In current work on ("reference") AUTOMATH one speaks about the "reference structure" of a correct book in order to isolate the analogue of what we called here "reference order".This way of speaking is rather informal within the AUTOMATH research group and the fact that correct books can be viewed as posets belongs,actually,to the "folklore" of the subject.

One should also note that the "reference structure" of correct AUT-books is to be distinguished carefully from both its "block structure" - describing the nesting of lines in correct books according to their contexts - and the "paragraph system structure",which is a local (practical) device,invented by I. Zandleven in order to simplify the actual checking of AUT-texts on a computer (cf. ZANDLEVEN 77-47 or JUTTING 79-46,Appendix 2.The paragraph system.)

Let us now examine the structure of reduced sites with respect to the reference order they contain.

A reduced LA-site B (for some k,with k in B) is certainly a poset,for it is,by definition,LA-compatible (see 321.2.).

It should be also clear that any LA-compatible site can be analysed into a finite number of reduced LA-sites.

There is,however,some more information we can obtain by tree-analysis in the case of a reduced site and this will be useful for the understanding of the "real structure" of an LA-compatible site,in general.

322. Proof theory: correctness proofs and their indices.

In this section we shall briefly examine some (neglected) aspects of the proof theory of the abstract AUT-languages of concern here. In particular, this *détour* serves to prove our key-corollary 322.34. below and is a reaction to criticism of L.S.van Benthem Jutting concerning a previous, elliptic treatement of the subject.
A more detailed discussion of the topics will be given elsewhere.

322.1. DEFINITION.

(1) A (*formal*) *correctness clause* (*in* LA) is an epi-statement of either one of the following forms:

(1°) $\mathbb{I}\ B;\Delta_0 \vdash_{LA} \tau\ \mathbb{I}$
read: "B is LA-compatible",

(2°) $\mathbb{I}\ B;\Delta \vdash_{LA} \tau\ \mathbb{I}$
read: "Δ is LA-admissible for B" (provided B is LA-compatible),

(3°) $\mathbb{I}\ (B,k);\Delta_0 \vdash_{LA} \tau\ \mathbb{I}$,
read: "$B\cup\{k\}$ is LA-compatible" (provided k is not in B),

(4°) $\mathbb{I}\ B;\Delta \vdash_{LA} a:b\ \mathbb{I}$

and

(5°) $\mathbb{I}\ B;\Delta \vdash_{LA} a\ \mathbb{I}$
with the usual reading (see, e.g., 20.4., provided B is LA-compatible and Δ is LA-admissible for B).

In the above, as expected, B is an LA-site, Δ is an LA-context, k is an LA-construction and a, b are LA-terms.

(2) The *initial correctness clause* (= *the initialization clause*) is a correctness clause of the form

$$\mathbb{I}\ \emptyset;\Delta_0 \vdash_{LA} \tau\ \mathbb{I}.$$

(3) A *correctness proof in* LA is a (finite) sequence $\bar{s}$ of correctness clauses in LA such that, for all s_i in $\bar{s}$, s_i is either

(1°) initial

or

(2°) follows from correctness clauses preceeding in the sequence, by applications of the (primitive) correctness rules of LA.

(4) A correctness proof in LA will be said to be a correctness proof of its last (correctness) clause.

(The subscripts "LA" will be omitted whenever no confusion is likely.)

322.2.REMARK.

The meaning of "follows from" in 322.1.(3) above can be made precise upon a suitable formalization of the epi-theory. We shall,however,skip the formal details,relying on the intuitions of the reader.(But see 322.5. below.)

322.3.REMARK.

Apparently,the typology of correctness clauses in 322.1. is incomplete,for the "derived" correctness rules of every LA - at least as stated here - contain also epi-statements of the form

"$B,\Delta \vdash_{LA} a \ \underline{\underline{R}} \ b$",

where $\underline{\underline{R}}$:= $\underline{\underline{contr}}$,$\underline{\underline{red}}$ or $\underline{\underline{conv}}$,while epi-statements of the form

"k is (sound) in/for B" (i.e., "$k \in B$"),

for LA-compatible B's and LA-constructions k,occur already in the primitive correctness rule (Er),which is a rule of PA,CA,etc. Accurately,these epi-statements can be "interpreted" in terms of (or "can be reduced to") the formal correctness clauses listed above and additional epi-theoretic constructs which remained "unformalized" so far.

Namely,

(1) where $\underline{\underline{R}}$ is as above,an epi-statement of the form

"$B,\Delta \vdash_{LA} a \ \underline{\underline{R}} \ b$"

is to be "interpreted" as the conjunction of

"$B,\Delta \vdash_{LA} a$" and "$B, \vdash_{LA} b$",

provided that a $\underline{\underline{R}}$ b holds in the associated reduction system.

(2) The epi-statements of the form

"k is (sound) in B"

are,in fact,disguised statements about correctness proofs in LA and we shall reserve them a special treatement below.

Note first the following trivial fact.

322.4.LEMMA.

Let B be an LA-compatible site.Then,for all k in B there are LA-compatible sub-sites $B_{(k+)}$ and $B_{(k-)}$ of B such that

(1) $B_{(k-)}$ does not contain k,

(2) $B_{(k+)} = B_{(k-)} \cup \{k\}$,

and,moreover,any correctness proof of

(1°) $[\![B_{(k+)};\Delta_0 \vdash \tau]\!]$

contains also a proof of

(2°) $[\![B_{(k-)};\Delta_0 \vdash \tau]\!]$

as a sub-proof (= sub-sequence of correctness clauses).

Proof.Indeed,in any correctness proof of (1°),k has to be introduced,sooner or later,by an application of a $\text{Site}^{LA}_{\boxtimes}$-recursion rule. That is: (1°) is proved on the basis of (2°) and,possibly,additional correctness clauses,by a single application of one of the rules (Sr-1p),(Sr-2p),(Sr-1d) or (Sr-2d).⊠

322.5.REMARKS.

It seems necessary to be more definite at this point.

(1) Let B be an LA-compatible site with k in B.Then,for every correctness proof $\bar{s}$ of

(1°) $[\![B;\Delta_0 \vdash \tau]\!]$,

$\bar{s}$ contains a correctness clause

(2°) $[\![(B^!_k,k);\Delta_0 \vdash \tau]\!]$

and a correctness clause

(3°) $[\![B^!_k;\Delta_0 \vdash \tau]\!]$

preceeding (2°) in $\bar{s}$,such that $B^!_k$ is LA-compatible,with

(4°) $B^!_k$ a subset of B

and

(5°) k not in $B^!_k$.

Moreover,$B^!_k$ is uniquely determined in/for any correctness proof $\bar{s}$ in LA (and any given k in B),provided $\bar{s}$ proves (1°). In the sequel,$B^!_k$ will be referred to as being the companion site of k in $\bar{s}$.

(2) If an epi-statement of the form

"k is (sound) in B"

is actually used at stage m in some correctness proof $\bar{s}$ in LA, this means that there are correctness clauses

(1°) $[\![(B^!_k,k);\Delta_0 \vdash \tau]\!]$,

(2°) $[\![B^!_k;\underline{\underline{ctx}}(k) \vdash \tau]\!]$,

(3°) $[\![B^!_k;\underline{\underline{ctx}}(k) \vdash \underline{\underline{cat}}(k)]\!]$

(resp.

(3°°) $[\![B^!_k;\underline{\underline{ctx}}(k) \vdash \underline{\underline{def}}(k) : \underline{\underline{cat}}(k)]\!]$

if k is a d-construction in LA), occurring at stages m_1, m_2, m_3 (resp. $m^!_3$) < m, together with a correctness clause

($°) $[\![(B^!_k,k);\Delta_0 \vdash \tau]\!]$

occurring at stage n in $\overline{s}$, with $m_1, m_2, m_3, (m^!_3) < n < m$, (where $B^!_k$ is the companion site of k in $\overline{s}$, while the "stages" of $\overline{s}$ refer, obviously, to lengths of initial segments of $\overline{s}$).

(2) In a more formal setting, it will be convenient to make always explicit <u>the actual uses</u> of the epi-statements of the form

(⊠) "k is (sound) in B"

in correctness proofs (in LA). In order to do this in a proper way we make the following <u>decisions</u>:

(1°) the epi-statements of the form (⊠) above will be viewed as <u>formal correctness clauses in</u> LA and,

(2°) upon adopting this point of view, we have to appropriately <u>extend the concept of a correctness proof in</u> LA. by

- letting the new formal clauses occur in correctness proofs (in some extended sense)

and by

- extending the meaning of "follows from" (in 322.1.) by supplementing the list of (primitive) correctness rules in every LA with (structural) rules for the manipulation (= introduction) of correctness clauses of the form (⊠).

322.6.DEFINITION.

(1) An epi-statement of the form

(6°) $[\![(k,B);\Delta_0 \vdash \tau]\!]$

(read: "k is (sound) in/for B"; provided B is LA-compatible - and k is actually an element of B -)

is a (<u>formal</u>) <u>correctness clause in</u> LA.

(2) An extended correctness proof in LA is defined as earlier, in 322.1., reading "correctness clause" in the present sense and including among the (primitive) correctness rules of (every) LA the structural rule (k) stated below.

(3) As earlier, an extended correctness proof is a proof of its last correctness clause.

322.7.COMMENT.

(1) Note that the extended correctness proofs are now allowed to have terminal clauses of the form

$$[\![\, (k,B);\Delta_0 \vdash \tau \,]\!],$$

which was not the case earlier.

(2) As we shall henceforth use only correctness proofs in the extended sense the specification "extended" will be, in general, omitted.

322.8.DEFINITION.

Let $\bar{s}$ be a correctness proof in LA. If B is an LA-site and k is an LA-construction we say that

a (Sr)-rule is applied to B and k at stage n in $\bar{s}$

if

(1°) the length of $\bar{s}$, $\underline{\underline{lh}}(\bar{s})$, is such that $\underline{\underline{lh}}(\bar{s}) \geqslant n$ $(n > 1)$,

(2°) $\bar{s}$ contains correctness clauses

$s_1 := [\![\, B;\Delta_0 \vdash \tau \,]\!]$, at stage n_1,

$s_2 := [\![\, B;\underline{\underline{ctx}}(k) \vdash \tau \,]\!]$, at stage n_2,

$s_3 := [\![\, B;\underline{\underline{ctx}}(k) \vdash \underline{\underline{cat}}(k) \,]\!]$, at stage n_3,

(and

$s_3^! := [\![\, B;\underline{\underline{ctx}}(k) \vdash \underline{\underline{def}}(k) : \underline{\underline{cat}}(k) \,]\!]$, at stage $n_3^!$,

whenever k is a d-construction),

with $n_1, n_2, n_3, (n_3^!) < n$, together with a correctness clause

$s := [\![\, (B,k),\Delta_0 \vdash \tau \,]\!]$, at stage n.

(Here $B = B_k^!$; so k is not in B and s is actually obtained from the s_i's by an application of a $\mathrm{Site}_{\boxtimes}^{LA}$-recursion rule. In other words, the disambiguating conditions concerning the freshness of floating constants occurring in the statement of these rules are supposed to hold in every case of concern.)

Now we can state the "structural" rules for the introduction of the (new) correctness clauses "$[\![(k,B);\Delta_0 \vdash \tau]\!]$" and the elimination of the correctness clauses "$[\![(B,k);\Delta_0 \vdash \tau]\!]$".
The terminology and notation are as in 322.8.Hereafter,$\bar{s}$ is a given correctness proof in LA,sufficiently long in order to make the stage-specifications appearing below meaningful.

STRUCTURAL RULES (k):

RULE (k-): If a (Sr-)rule is applied to B' and k at some stage n in $\bar{s}$ then,for $B := B' \cup \{k\}$, the clause $[\![B;\Delta_0 \vdash \tau]\!]$ is allowed to occur in $\bar{s}$ at any stage m,m $>$ n.

RULE (k+): If

1° a (Sr-)rule is applied to B' and k at some stage n in $\bar{s}$ and,for some B with $B' \sqsubseteq B$,

2° $[\![B;\Delta_0 \vdash_{LA} \tau]\!]$

and

3° $[\![B;\underline{\underline{ctx}}(k) \vdash_{LA} \tau]\!]$

occur in $\bar{s}$ at stages m_1,m_2 resp.,with $n \leqslant m_1,m_2$,

then

4° $[\![(k,B);\Delta_0 \vdash_{LA} \tau]\!]$

is allowed to occur at any stage m in $\bar{s}$,with $n,m_1,m_2 < m$.

322.9.REMARKS.

(1) In the context of the remaining (primitive) correctness rules of LA,the structural rule (k+) says,roughly speaking,that a correctness clause

$$[\![(k,B);\Delta_0 \vdash \tau]\!]$$

can be introduced,in any correctness proof $\bar{s}$ in LA,on the basis of a previous application of one of the $\text{Site}_{\boxtimes}^{LA}$-recursion rules and only in these conditions.

(2) The foregoing considerations should make now precise enough the content of the $\text{Esent}_{\boxtimes}^{LA}$-recursion rule (Er),the only primitive correctness rule of PA,CA,etc. containing epi-statements of the form

"k is (sound) in B".

Specifically,in (meta-)proofs by induction on correctness in LA (= proofs "by induction on the length of a correctness proof in LA"), one has to keep in mind that any application of (Er) at

some stage n in some correctness proof $\bar{s}$ is accompanied by a previous application of the relevant $Site^{LA}_{\boxtimes}$- recursion rule at some stage m, with $m < n$, in $\bar{s}$.
Concurrently, if some induction concerns (Er) "at stage n" then the hypothesis of the induction concerns (k+) "at any stage m", with $m < n$, where (k+) involves a "nested" inductive step.
(3) It should be clear that the present considerations (and, in particular, the addition of rules (k) to the primitive apparatus of the LA's of concern here) do not actually increase the "deductive strength" of the original formulations, but rather make them more explicit. In particular, (k-) has only a notational effect.

322.10.REMARK. (Structured correctness proofs.)
We shall unessentially modify, once more, the concept of a correctness proof in LA in order to economize on later considerations about ranking functions (see 101.1.).
Let us replace "rank-fresh" by "minimally fresh" (see 122.11. for definitions), everywhere in the $Site^{LA}_{\boxtimes}$-recursion rules. Call the correctness proofs obtained by this restriction <u>structured</u> <u>correctness</u> <u>proofs</u>. One realizes easily that a structured proof (in LA) is still a correctness proof in LA (in the original, unrestricted sense; cf. also 122.12.). Clearly, sub-proofs (i.e., sub-sequences that are correctness proofs) of a structured proof in LA are also structured. Moreover, there is no loss in generality if we choose to practice always a "structured" way of proving correctness in LA. Indeed, the differences thereby induced will only consist of a uniform ("isomorphic" say) relettering of the floating constants used in particular LA-compatible sites. (This is, in the end, equivalent to the choice of a different ranking function for the floating constants in the alphabet of LA. Cf. 101.1.).

322.11.REMARK.NOTATION.
Let B be an LA-compatible site. Recall that, according to 101.8., <u>the</u> <u>rank</u> <u>of</u> B has to be

$$\mathrm{rank}(B) = \max\{\mathrm{rank}(\mathrm{idf}(k)) : k \in B\}.$$

So, for any k in B, $\mathrm{rank}(\mathrm{idf}(k)) \leqslant \mathrm{rank}(B)$.

We shall now introduce an inductive "measure of complexity" for LA-constructions occurring in LA-compatible sites using the reference order introduced earlier. This will turn out to have the same effect as the manipulation of ranks in structured proofs of correctness (in LA).

322.12. DEFINITION.

Let B be an LA-compatible site with k in B.

(1) One defines _the depth of k in B_ (notation: $\mathrm{depth}_B(k)$) as follows:

(1°) if, for no k' in B,

$$k' \prec k,$$

then $\mathrm{depth}_B(k) = 0$;

(2°) if for $n \geq 1$, and $k_1, \ldots, k_n$ in B

$$k_1, \ldots, k_n \prec k,$$

(and there is no other k' in B with this property) then

$$\mathrm{depth}_B(k) = \max\{\mathrm{depth}_B(k_i) : 1 \leq i \leq n\} + 1.$$

(2) _The depth of_ B (notation: $\mathrm{depth}(B)$) is then

$$\mathrm{depth}(B) = \max\{\mathrm{depth}_B(k) : k \in B\}.$$

322.13. DEFINITION.

Let B be an LA-compatible site with $\mathrm{depth}(B) = m$. Then, for all n, $0 \leq n \leq m$,

(1) _the layer of depth_ n _in_ B is the site

$$\hat{L}(B,n) = \{k \in B : \mathrm{depth}_B(k) = n\}.$$

and

(2) _the section of depth_ n _in_ B (_the_ n-_section of_ B) is the site

$$B^{\$}_n = \bigcup_{0 \leq i \leq n} \hat{L}(B,i).$$

(For convenience, we set also $B^{\$}_{-1} = \emptyset$.)

322.14. COMMENT.

Intuitively, one may think of the depth of an LA-construction/site as being a kind of "invariance property" of the construction/site under consideration, relative to the underlying reference order of the site.

The rôle of the structured proofs will become now obvious.

322.15.LEMMA.

Let B be an LA-compatible site.Then every structured proof of

$$[\![\, B;\Delta_0 \vdash_{LA} \tau \,]\!]$$

gives also

$$\mathrm{rank}(\mathrm{idf}(k) \;=\; \mathrm{depth}_B(k)$$

and,consequently,

$$\mathrm{rank}(B) \;=\; \mathrm{depth}(B).$$

Proof.By induction on correctness in LA.⊠

322.16.REMARK.

Let B be an LA-compatible site,with k_1,k_2 in B.If

(1^o) $\quad k_1 \prec k_2$

then (clearly,$k_1 \neq k_2$ and) every structured proof of

$$[\![\, B;\Delta_0 \vdash_{LA} \tau \,]\!]$$

gives also

(2^o) $\quad \mathrm{rank}(\mathrm{idf}(k_1) < \mathrm{rank}(\mathrm{idf}(k_2))$

for (1^o) implies already that

(3^o) $\quad \mathrm{depth}_B(k_1) < \mathrm{depth}_B(k_2)$.

A similar fact holds,*mutatis mutandis*,for $\preccurlyeq$ and $\leqslant$.
However,these implications cannot be reversed,in general,for one can have (3^o) say,without thereby having k_1 and k_2 comparable relative to the reference order of B.

The following set-theoretic ingredient will be useful later.

322.17.THEOREM.("Union Theorem".)

Let B be an LA-compatible site.If B_1,B_2 are LA-compatible subsets of B then so is their union.

Proof.Easy.(*Hint*: use Site-expansion results discussed in 213.Cf. also 222.20.The rest is a typical question of "recreational mathematics".)⊠

322.18.COROLLARY.

(1) Any (finite) union of LA-compatible sites (resp. reduced subsites) of an LA-compatible site is,again,LA-compatible.

(2) Every LA-compatible site is a (finite) union of LA-compatible (and,in particular,reduced) sites.

Proof.(1) By successive applications of 322.17.
(2) Take,e.g.,the union of the sets $B_k^v \sqsubseteq B$,such that B_k^v is reduced for k,with k in B.⊠

322.19.REMARK.

In general,$Site_{⊠}^{LA}$ is not closed under finite unions,for the identifiers of LA-constructions "occurring in" distinct LA-compatible sites need not be (pairwise) distinct.But,if we assume that $\underline{\underline{idf}}(B_i)$,$1 \leq i \leq n$,are pairwise disjoint sets whenever so are the B_i's,then the union of the B_i's is actually LA-compatible if so is each B_i $(1 \leq i \leq n)$.(The first part of the remark is due to L.S.van Benthem Jutting,in conversation.)

322.20.NOTATION.

Let X be some wfe "occurring in" some LA-compatible site B and $\underline{\underline{T}}(X)$ be the analytic tree of X (in B).Then we let $\underline{\underline{ST}}(X)$ be the set of subtrees $\underline{\underline{H}}$ of $\underline{\underline{ST}}(X)$ such that $\underline{\underline{H}}$ is a construction-tree.(Note that one may have $\underline{\underline{H}} = \underline{\underline{T}}(X)$,as well.)

322.21.DEFINITION.

Let X be a wfe "occurring in" some LA-compatible site B.Then the analytic site (analytic set or a-set) of X (in B) is the site

$$\underline{\underline{CT}}(X) = \{k \in Constr^{LA}: \ \underline{\underline{T}}(k) \text{ is in } \underline{\underline{ST}}(X)\}.$$

322.22.COMMENT.

That is: given some LA-construction (LA-context,E-sentence or term in LA) X say,we can always obtain the analytic site of X in an effective way,by a mere "syntactic" analysis of X into its (both immediate and remote) "components",provided X "occurs in" some LA-compatible site B.The "algorithm" is,indeed,straightforward: construct first the analytic tree of X (this is just "syntactic" analysis;hence the name above),then collect the labels of the nodes of this tree and select only those labels that are pairs of the form $\langle k_{\underline{c}},\underline{c}\rangle$.The first projection of any such a pair is an LA-construction on which X actually "depends", in the sense of the reference order of B.The set of all these LA-constructions is the analytic site of X (in B).

323.23.REMARK.

(1) If B is LA-compatible and k is in B then

(1°) $\underline{\underline{CT}}(k) = \{k' \in B: k' \preccurlyeq k\}$

and

(2°) $\underline{\underline{CT}}(k) \sqsubseteq B$.

(2) It is possible to define analogously "genetic sites" of LA-constructions, LA-contexts, E-sentences, etc. "occurring in" some LA-compatible site B, replacing the analytic trees in the definition above by genetic trees (cf. 313.). The "genetic sites" will, however, depend on particular proofs of correctness for the LA-site B and, consequently, the corresponding analysis will be "intensional" from the very beginning .

We shall not pursue the investigation on these lines (i.e., using "genetic sites"), but will employ a different technique, also "intensional" in nature, closely related to the "encoding" techniques of recursion theory.

The plan behind what follows consists of "encoding" directly correctness clauses (resp. structured correctness proofs) in LA on finite unions of analytic sites. This will be done by an appropriate "indexing" of the (correctness) clauses occurring in (structured) correctness proofs in LA. The procedure will reflect in some straightforward sense the "construction" of analytic sites, and seems to be of interest beyond the particular purposes of this section.

322.24.NOTATION.

Let $\bar{s}$ be a (structured) correctness proof in LA. The correctness index of a (correctness) clause

$$s⟦ B ⟧ := ⟦ \ldots B \ldots ⟧$$

of $\bar{s}$ will be defined in 322.25. Here and everywhere in the sequel we adopt the following notational conventions:

Let B be an LA-compatible site.

(1) Where ⟦ ... B ... ⟧ is a correctness clause "containing" B, $\underline{\underline{idx}}$ ⟦ ... B ... ⟧ is its correctness index.

(2) Where B' is any other LA-site,

$$s\llbracket B' \rrbracket := \llbracket \ldots B' \ldots \rrbracket$$

and

$$\underline{\underline{idx}} \llbracket \ldots B' \ldots \rrbracket$$

resp. stand for the fact that B has been replaced, *ceteribus paribus*, with B', in the corresponding correctness clause. (This is, accurately, a "substitution operation" defined *on* the meta-language and, clearly, can be introduced by an obvious formal definition, upon a suitable "advanced" formalization of the epi-theory.)

(3) We use also the following shorthand:

$$\underline{\underline{idx}} \llbracket B;\Delta \vdash a_1,\ldots,a_n \rrbracket = \bigcup_{1\leq i\leq n} \underline{\underline{idx}} \llbracket B;\Delta \vdash a_i \rrbracket.$$

(4) If k is an LA-construction (in B) then

$$\underline{\underline{CT}}_{\boxtimes}(k) = \begin{cases} \underline{\underline{idx}} \llbracket B;\underline{\underline{ctx}}(k) \vdash \tau,\underline{\underline{cat}}(k) \rrbracket, \text{if k is in } Pconstr^{LA} \\ \\ \underline{\underline{idx}} \llbracket B;\underline{\underline{ctx}}(k) \vdash \tau,\underline{\underline{cat}}(k),\underline{\underline{def}}(k) \rrbracket, \text{else.} \end{cases}$$

(5) If Δ is LA-admissible for B and a is a head-term with

$$B,\Delta \vdash a,$$

then $\underline{\underline{lh}}(\Delta) = \underline{\underline{lh}}(\underline{\underline{tail}}(a)) = n, n \geqslant 0$, and CAT(a) will denote *the unique* k in B with

(1°) $\underline{\underline{idf}}(k) \equiv \underline{\underline{head}}(a)$

and

(2°) $\underline{\underline{lh}}(k) = n$.

(See 212.12., 212.13. and 222.8. through 222.10. above.)

(6) As earlier, $B^!_k$ is the companion site of k in $\overline{s}$.

322.25. DEFINITION. (Correctness indices.)

(1) With notation as in 322.14., $\underline{\underline{idx}} \llbracket \ldots B \ldots \rrbracket$ is defined by cases as follows (here: B is an LA-compatible site).

(1°) $\underline{\underline{idx}} \llbracket B;\Delta_0 \vdash \tau \rrbracket = \emptyset$;

(2°) $\underline{\underline{idx}} \llbracket (B,k);\Delta_0 \vdash \tau \rrbracket = \underline{\underline{CT}}_{\boxtimes}(k)$, provided k is not in B;

(3°) $\underline{\underline{idx}} \llbracket (k,B);\Delta_0 \vdash \tau \rrbracket = \underline{\underline{idx}} \llbracket (B^!_k,k);\Delta_0 \vdash \tau \rrbracket \cup \{k\}$;
provided k is in B;

(4°) $\underline{\underline{idx}}\,[\![\, B;\Delta[v:a] \vdash \tau \,]\!] = \underline{\underline{idx}}\,[\![\, B;\Delta \vdash \tau, a \,]\!]$;

(5°) $\underline{\underline{idx}}\,[\![\, B;\Delta \vdash a{:}b \,]\!]$ is, by induction on the structure of a,

$= \underline{\underline{ctx}}\,[\![\, B;\Delta \vdash \tau \,]\!]$, if a is a variable

$= \underline{\underline{idx}}\,[\![\, B;\Delta \vdash \tau, \underline{\underline{arm}}_1^n(a), \ldots, \underline{\underline{arm}}_n^n(a) \,]\!] \cup \underline{\underline{CT}}_{\boxtimes}(k) \cup \{k\}$,

if a is a head-term with $k \equiv CAT(a)$ and $n = \underline{\underline{lh}}(k) = \underline{\underline{lh}}(\underline{\underline{ctx}}(k))$;

$= \underline{\underline{idx}}\,[\![\, B;\Delta \vdash \tau, \underline{\underline{arg}}(a), \underline{\underline{fun}}(a) \,]\!]$,

if a is an application term;

$= \underline{\underline{idx}}\,[\![\, B;\Delta \vdash \tau, \underline{\underline{dom}}(a), \underline{\underline{val}}(a) \,]\!]$,

if a is an abstraction term;

(6°) $\underline{\underline{idx}}\,[\![\, B;\Delta \vdash a \,]\!] = \begin{cases} \underline{\underline{idx}}\,[\![\, B;\Delta \vdash \tau \,]\!], \text{ if } a \equiv \tau, (PA,CA) \\ \underline{\underline{idx}}\,(\, B;\Delta \vdash \tau, a_1, \ldots, a_n \,), \\ \quad \text{if } a \equiv [v_1{:}a_1] \ldots [v_n{:}a_n]\tau, (Q^-A, QA) \\ \underline{\underline{idx}}\,[\![\, B;\Delta \vdash a{:}b \,]\!], \text{else,} \\ \quad \text{where b is an LA-term with } B,\Delta \vdash_{LA} a{:}b. \end{cases}$

(2) The (correctness) index of the proof $\bar{s}$ (in LA) is the index of the last correctness clause of $\bar{s}$.

322.26. REMARK.

If $\underline{\underline{R}} := \underline{\underline{contr}}, \underline{\underline{red}}, \underline{\underline{conv}}$ and one wants to formalize the LA's as indicated in 10.1. above (with $\underline{C}, \underline{R}, \underline{Q}$ as primitive relators) then one should have correctness clauses corresponding to epi-statements of the form

$$(\boxtimes) \qquad B, \Delta \models a \;\underline{\underline{R}}\; b.$$

In such cases, we might have defined the indices of the corresponding (formal) correctness clauses as being

$$\underline{\underline{idx}}\,[\![\, B;\Delta \vdash \tau, a, b \,]\!].$$

In the present setting, the indices are also supposed to remain invariant under contraction (resp. reduction, conversion) in the associated reduction system. (In fact, correctness indices remain primitively undefined for epi-statements of the form ($\boxtimes$) above, since such statements have no "formal" counterpart. So the presupposition of invariance is, actually, tacit.)

322.27.REMARK.

If $s\llbracket B \rrbracket := \llbracket \ldots B \ldots \rrbracket$ is a correctness clause occurring in some (structured) correctness proof $\bar{s}$ (and B is LA-compatible) then $\underline{\underline{idx}}(s\llbracket B \rrbracket)$ is an LA-site and a subset of B.

322.28.LEMMA.

Let $\bar{s}$ be a (structured) correctness proof in LA,Where B is an LA-compatible site, $\Delta := [v_1:a_1]\ldots[v_n:a_n]$ is an LA-context with $\underline{\underline{lh}}(\Delta) = n, n \geq 0$,k is an LA-construction and a,b are LA-terms one has:

(1) If

$$s_1 := \llbracket B; \Delta \vdash \tau \rrbracket$$

occurs in $\bar{s}$ then

$$\underline{\underline{idx}}(s_1) = \underline{\underline{idx}} \llbracket B; \Delta_0 \vdash \tau, a_1, \ldots, a_n \rrbracket.$$

(2) If

$$s_2 := \llbracket B; \Delta \vdash a \rrbracket$$

occurs in $\bar{s}$ then s_1 does and

$$\underline{\underline{idx}}(s_2) = \underline{\underline{idx}}(s_1) \cup \underline{\underline{CT}}(a).$$

(3) Thus,in the same conditions

$$\underline{\underline{idx}}(s_1) = \underline{\underline{CT}}(\Delta)$$

and

$$\underline{\underline{idx}}(s_2) = \underline{\underline{CT}}(\Delta) \cup \underline{\underline{CT}}(a).$$

(4) Moreover,for some LA-term b,

$$s_3 := \llbracket B; \Delta \vdash a:b \rrbracket$$

occurs in $\bar{s}$ if s_2 does and

$$\underline{\underline{idx}}(s_3) = \underline{\underline{idx}}(s_2).$$

Finally,

(5) if

$$s_k^- := \llbracket (B,k); \Delta_0 \vdash \tau \rrbracket$$

occurs in $\bar{s}$ then

$$\underline{\underline{idx}}(s_k^-) = \underline{\underline{CT}}(k) - \{k\},$$

and

(6) if

$$s_k^+ := \llbracket (k,B); \Delta_0 \vdash \tau \rrbracket$$

occurs in $\bar{s}$ then

$$\underline{\underline{idx}}(s_k^+) = \underline{\underline{CT}}(k).$$

Proof.(1) is straightforward.(2),(5) and (6) follow by easy inductions.Now (3) is a consequence of (1) and (2),while (4) is clear from the definition of $\underline{\underline{\mathrm{idx}}}$.$\boxtimes$

322.29.COMMENT.

So the correctness index of a clause occurring in some (structured) correctness proof $\overline{s}$ in LA is always either

(1) an analytic site or

(2) a (finite) union of analytic sites.

Now we are able to prove the following "reflection property" of the correctness indices.

322.30.THEOREM.("Index Theorem".)

Let B be an LA-compatible site and $\overline{s}(B)$ be a (structured) correctness proof of

$$\llbracket B;\Delta_0 \vdash \tau \rrbracket.$$

Then

(1) for all correctness clauses

$$s\Pi\ B_{\boxtimes}\ \Pi := \llbracket \ldots B_{\boxtimes} \ldots \rrbracket$$

occurring in $\overline{s}$ (and "containing" an LA-compatible sub-site $B_{\boxtimes} \sqsubseteq B$)

(11) one can find (effectively),for every LA-compatible site B_i,with

$$\underline{\underline{\mathrm{idx}}}\ \llbracket \ldots B_{\boxtimes} \ldots \rrbracket \sqsubseteq B_i \sqsubseteq B_{\boxtimes},$$

a (correctness) proof $\hat{s}(B_i)$ of

$$s\Pi\ B_i\ \Pi := \llbracket \ldots B_i \ldots \rrbracket$$

where

(12) $\hat{s}(B_i)$ contains (as a sub-proof) a proof of

$$\llbracket B_i;\Delta_0 \vdash \tau \rrbracket.$$

(2) In particular,for all correctness clauses $s\Pi\ B_{\boxtimes}\ \Pi$ of $\overline{s}$,

(21) there is a proof $\hat{s}$ of

$$s\Pi\ \underline{\underline{\mathrm{idx}}}(s\Pi\ B_{\boxtimes}\ \Pi)\ \Pi := \llbracket \ldots \underline{\underline{\mathrm{idx}}}\ \llbracket \ldots B_{\boxtimes} \ldots \rrbracket \ldots \rrbracket$$

such that

(22) $\hat{s}$ contains (as a sub-proof) a proof of

$$\llbracket \underline{\underline{\mathrm{idx}}}(s\Pi\ B_{\boxtimes}\ \Pi);\Delta_0 \vdash \tau \rrbracket$$

with,moreover,

(23) $$\underline{\underline{\mathrm{idx}}}\ \llbracket \ldots B_{\boxtimes} \ldots \rrbracket = \underline{\underline{\mathrm{idx}}}\ \llbracket \ldots \underline{\underline{\mathrm{idx}}}\ \llbracket \ldots B_{\boxtimes} \ldots \rrbracket \ldots \rrbracket$$

Proof.(I) Note first that with "effectively" in (11), the method of constructing the required $\hat{s}(B_i)$'s in (1) does already guarantee an effective proof of (21) and (22), which are particular cases of (11) and (12) resp. In fact, there is a (nearly mechanical) procedure of "expanding" the given correctness proof $\bar{s}$ up to the required $\hat{s}(B_i)$'s, from which we can "read off" (23) directly. The construction is rather tedious and requires some amount of inventivity in finding an appropriate system of notation for "proof-analyses" (i.e., the indications concerning the way a correctness rule of LA is actually applied to clauses of $\bar{s}$ in order to produce other clauses of $\bar{s}$).
A detailed discussion of the method (of finding "complete expansions of a given correctness proof in LA") is deferred and will appear elsewhere. Here, we have to proceed somewhat differently, since the main interest is, in fact, in the "partial" case, stated in (2).
(II) Neglecting "effectively" in (1) has the effect of commiting us to prove (2) first. Note that, actually, (1) follows (without effectiveness, however) from (21) and (22) by results on $\mathrm{Site}^{LA}_{\boxtimes}$-expansion discussed in 213. (and, incidentally, by use of 322.17. and 322.18.(1) above).
Now the proof of (2) is by "induction on correctness in LA", in the formal setting introduced in this section. Specifically, one has three "properties" of the formal correctness clauses of LA, viz.

(1°) "derivability (-invariance) under self-index replacement", (stated in (21)),

(2°) "LA-compatibility of correctness indices" (22)

and

(3°) "fixed-point condition" for indices (23).

Clearly, the initialization clause

$$[\![\emptyset;\Delta_0 \vdash \tau]\!]$$

satisfies all these properties and it is an easy (though somewhat tedious) exercise to show that the correctness rules of every LA (including the "structural" rules (k), stated above) preserve these properties. (Note also that (2°) requires the use of the "Union Theorem", specifically: 322.18.(1) above.). Obviously, the "modifying" rules (CC_i), i = 1,2, and - in the case of QA - (CI) are harmless for the purposes of the proof. (Cf. 322.25. again.). Lemma 322.28. has to be used copiusly at the inductive step, of course. ⊠

322.31.COMMENT.

The specific part (2) of Theorem 322.30. says,roughly speaking, that:

(1) if something can be proved ("in" LA) at all,of some LA-compatible site $B_{\boxtimes}$ then the same thing can be shown to hold of/for the correctness index of the corresponding proof;

(2) in particular,the correctness index of a (correctness) proof in LA is always an LA-compatible site

and

(3) the correctness index of a given clause s,occurring in some (correctness) proof $\overline{s}$ (in LA),is the "minimal" LA-compatible site of which s holds ("fixed-point condition").

More or less metaphorically,one can say that the correctness indices are "fixed points" of correctness proofs in LA.(But,accurately,the "epi-theoretic function(s)"

(£) $\lambda x.\underline{idx}\ [\ \ldots\ x\ \ldots\ ]$

has/have correctness indices as fixed points only if it is/they are "correctly parametrized".Indeed,"epi-functional expressions" of the form (£) above may also contain "hidden parameters" as, e.g.,LA-contexts,LA-constructions or LA-terms.).

For the sake of clarity we extract from 322.30. only the information neccesary for the proof of 322.34.

322.32.COROLLARY.

Let $\overline{s}$ be a (structured) correctness proof in LA.If s is a correctness clause of $\overline{s}$ then $\underline{idx}$(s) is an LA-compatible site.

Proof.This is the content of 322.30.(22).$\boxtimes$

322.33.REMARK.

Let B be an LA-compatible site and k be an element of B.Where $\underline{depth}_B(k) = n$ and $B^!_k$ is,as above,the companion site of k in some (correctness) proof $\overline{s}$ of

$$[\ B;\Delta_0 \vdash \tau\ ],$$

it is clear that $\overline{s}$ should contain (as a sub-proof) a proof $\hat{s}(k)$ of

$$[\ (B^!_k,k);\Delta_0 \vdash \tau\ ] \qquad \text{(k is not in } B^!_k\text{)}$$

with correctness index $B(k) = \underline{CT}(k) - \{k\}$.

Moreover, one has easily that

(1°) $B(k) = \emptyset$, if $n = 0$,

and

(2°) $B(k) \subseteq B^{\$}_{n-1}$, if $n \geqslant 1$.

In fact,

(3°) $\underline{\underline{depth}}(B(k)) = \underline{\underline{depth}}_B(k) + 1$

(where, by convention, $\underline{\underline{depth}}(\emptyset) = -1$; cf. also 322.13.(2) above) and $B(k)$ need not be the whole section $B^{\$}_{n-1}$ (when $n \neq 0$). Indeed, in general, $B(k)$ is a <u>proper</u> subset of the (n-1)-section unless B is a reduced site. (See below why.)

Finally, note that

(4°) if $k' \in \underline{\underline{CT}}(k)$ and $k' \neq k$ then $k' \in B(k)$.

322.34. COROLLARY.

Let B be an LA-compatible site with k in B. Then $\underline{\underline{CT}}(k)$ is an LA-compatible site.

<u>Proof</u>. Let $\overline{s}$ be a (structured correctness) proof of

$$[\![B; \Delta_0 \vdash \tau]\!].$$

As already noted, $\overline{s}$ should contain a proof of

$$[\![(B^!_k, k); \Delta_0 \vdash \tau]\!]$$

as a sub-proof. Assume that the latter clause occurs in $\overline{s}$ at stage n, $n > 1$. Then one can always insert, by applying rule (k+), a clause

$$[\![(k, B'); \Delta_0 \vdash \tau]\!]$$

at some stage m, $m > n$, in $\overline{s}$, where B' is some LA-compatible site with

$$B^!_k \cup \{k\} \subseteq B' \subseteq B.$$

(In particular, one can take $B' = B^!_k \cup \{k\}$, if the corresponding clause does not occur already in $\overline{s}$.)

Now

$$\underline{\underline{idx}} [\![(k, B'); \Delta_0 \vdash \tau]\!] = \underline{\underline{CT}}(k),$$

by 322.28.(6), and, by 322.32., indices of correctness clauses are LA-compatible sites. That is: $\underline{\underline{CT}}(k)$ is LA-compatible. ⊠

323. Analytic sites qua reducts and some topology.

The notion of a reduced site (cf. 321.1.) is a "global" concept and, in particular, it is not easily manipulated in actual reasoning on LA-compatible sites. Moreover, the way it was introduced is not very constructive. On the other hand, the analytic sites, discussed in 322. were found to be effective, but the considerations concerning their "minimality" were essentially dependent on particular correctness clauses occurring in correctness proofs in LA.
In what follows we establish the equivalence of these notions.

323.1. LEMMA. ("Separation".)

Let B be an LA-compatible site with k_1, k_2 in B ($k_1 \neq k_2$) such that

$$k_1 \prec k_2.$$

Then there is an LA-compatible site $B_{\boxtimes}$ such that $B_{\boxtimes} \sqsubseteq B$ and k_1 is in $B_{\boxtimes}$ but k_2 is not in $B_{\boxtimes}$.

Proof. Let $B_2^!$ be the companion site of k_2 in some correctness proof $\bar{s}$ of

$$[\![B; \Delta_0 \vdash \tau]\!].$$

Then, by 322.5., there is a proof $\bar{s}(k_2)$ of

$$s_2 := [\![(B_2^!, k_2); \Delta_0 \vdash \tau]\!].$$

Set $B_{\boxtimes} = \underline{idx}(s_2)$. This site satisfies the requirements since it is LA-compatible, by 322.32., and, moreover, $B_{\boxtimes} = \underline{CT}(k_2) - \{k_2\}$, by 322.28 (5). So $B_{\boxtimes}$ does not contain k_2. Now k_1 is in $B_{\boxtimes}$ by the definition of $\underline{CT}$. ⊠

323.2. COROLLARY.

Let B be a reduced site for k (k in B). Then k is maximal (relative to the reference order) in B.

Proof. By reductio and 323.1. ⊠

323.3. LEMMA.

Let B be a reduced site for k (k in B). Then B - $\{k\}$ is LA-compatible.

Proof. Use 322.4. and the fact that B is reduced for k. ⊠

323.4.COROLLARY.

Let B be a reduced site for k_1,k_2 (k_1,k_2 in B).Then $k_1 \equiv k_2$.
Proof.By reductio.If $k_1 \not\equiv k_2$ then k_1,k_2 are both maximal in B,by 323.2,and "extracting" each of them from B gives LA-compatible sites $B - \{k_1\}$ and $B - \{k_2\}$resp.This is impossible unless $k_1 \equiv k_2$.⊠

323.5.COMMENT.

So any reduced LA-site has a "supremum" with respect to its reference order.Specifically,if B is reduced for k then its "supremum" is k itself and we call it the bottom of B.(This terminology is somewhat artificial but it has something to do with our way of drawing the "graph" of a reduced site: the "supremum" will be always placed at the bottom of the picture representing the reduced site qua poset.).

We may interpret heuristically the reference order of an LA-compatible site B as a rough estimation of the "amount of information" conveyed by each LA-construction in B.Within this heuristics one would want to begin with "minimal pieces of information" (furnished by LA-constructions with depth 0 in B).An LA-construction k strictly depending - in the sense of the reference order - on LA-constructions $k_1,\dots,k_n$ ($n \geqslant 1$) in B will be then supposed to "contain more information" than the k_i's.In particular,if the site is reduced (for some k) then its "bottom" provides always the maximum of information for that site.

323.6.REMARKS.

(1) If B is a reduced site for some k (k in B) then,by 323.3. the depth of B equals the depth of k in B and any other k' in B has smaller depth (in B) than k itself.
(2) Every reduced site B is a semi-lattice (relative to the reference order in B).

323.7.LEMMA.

Let B be an LA-compatible site with k in B.Then CT(k),the analytic set of k is contained (qua set) in any LA-compatible subset B_k of B which has k as an element.

Proof. One shows easily, by a systematic search throughout the analytic tree of k that there is no k_o in B with $k_o \neq k$, $k_o \prec k$ and k_o in B - B_k.⊠

323.8. COROLLARY.

Let B_k be a reduced site for k (k in B). Then $\underline{\underline{CT}}(k) \sqsubseteq B_k$.

Proof. By definition, B_k is compatible and contains k. Then apply 323.7.⊠

323.9. THEOREM.

Let B be an LA-compatible site with k in B. Then, for some reduct B_k^v of B for k, one has $B_k^v = \underline{\underline{CT}}(k)$.

Proof. $\underline{\underline{CT}}(k)$ is an LA-compatible subset of B containing k as an element. Then $\underline{\underline{CT}}(k)$ contains a minimal subset B_k^v say, which is LA-compatible and has k as an element. So B_k^v is a subset of $\underline{\underline{CT}}(k)$. But $\underline{\underline{CT}}(k)$ is a subset of every reduct of B for k (by 323.8.). So, in particular, $B_k^v = \underline{\underline{CT}}(k)$.⊠

323.10. COROLLARY.

Let B be an LA-compatible site and k be in B. If $B_{k,1}^v$ and $B_{k,2}^v$ are reducts of B for k then $B_{k,1}^v = B_{k,2}^v$.

Proof. By 323.8. The sets $B_{k,i}^v$ (i = 1,2) contain (both) $\underline{\underline{CT}}(k)$. So, for some reduct $B_{k,0}^v$ of B for k, one has $\underline{\underline{CT}}(k) = B_{k,0}^v$, by 323.9., and $B_{k,0}^v$ is a reduct of B for k, contained in reducts $B_{k,1}^v$, $B_{k,2}^v$ of B for k. This is possible iff $B_{k,0}^v = B_{k,i}^v$ (i = 1,2).⊠

Explicitely, one has a

323.11. THEOREM.

(1) For each LA-compatible site B and every k in B, there is a unique LA-compatible subset B_k^v of B such that B_k^v is a reduced site for k (= there is a unique reduct of B for k).

(2) Moreover, the reduct of an LA-compatible site B for some k (k in B) can be obtained effectively from k (by "syntactic" analysis, *via* analytic trees), with specifically, $B_k^v = \underline{\underline{CT}}(k)$.

Proof. Already completed.⊠

323.12.COMMENT (historical).

The facts mentioned above have been conjectured - several years ago - by N.G.de Bruijn et al. in connection with the excerpt program for AUT-QE mentioned earlier (321.3.).The present analysis was,however,done in complete independence of previous work on that program and its implementation (at the Technische Hogeschool, Eindhoven).

As the foregoing analysis is a "global" one (viz.,it does not depend too much on the exact formulation of the correctness rules for some particular AUT-language) the results obtained here can be easily transferred to other AUT-languages,not considered here (whether in abstract or "reference" formulation).

It is also useful to insist,once more,on the set-theoretic behaviour of LA-compatible sub-sites of a given LA-compatible site.

323.13.THEOREM.("Intersection Theorem".)

Let B be an LA-compatible site.If B_1,B_2 are LA-compatible subsets of B then so is their intersection.

Proof.If $B_1 \sqcap B_2 = \emptyset$ then the Theorem follows by (Si).Else,let $B_1 \sqcap B_2 = \{k_1,\ldots,k_n\}$,with $n \geq 1$.Then,for every $j, 1 \leq j \leq n$, $\underline{\underline{CT}}(k_j)$ is a subset of both B_1 and B_2,by 323.7.,and $\underline{\underline{CT}}(k_j)$ is LA-compatible, by 322.34.Now $B_1 \sqcap B_2 = \bigsqcup_{1\leq j\leq n} \underline{\underline{CT}}(k_j)$,by the definition of $\underline{\underline{CT}}(k_j)$,and the latter set is LA-compatible,by 322.18.⊠

323.14.COROLLARY.

Let B be an LA-compatible site.Then any (finite) intersection of LA-compatible subsets of B is,again,LA-compatible.

Proof.By successive applications of 323.13.⊠

Note also the following

323.15.THEOREM.

Every section of an LA-compatible site is LA-compatible.

Proof.If the site is empty,use (Si).Else,realize that,for every LA-compatible B,with $\underline{\underline{depth}}(B) = n, n \geq 0, B_n^{\$}$ is the union of analytic sites $\underline{\underline{CT}}(k)$ with k in $B_n^{\$}$,and apply 322.18.⊠

Now we have some reason to shift to a topological terminology.

323.16.DEFINITION.

(1) An LA-space is an LA-compatible site.

(2) Let B be an LA-space.A subset of B is B-open (in LA) if it is an LA-compatible sub-site of B.

(3) If B is an LA-space then the natural topology on B is the family $\hat{T}(B) = \{B' : (B' \sqsubseteq B) \& (B' \text{ is B-open})\}$.

This way of speaking is motivated by the following.

323.17.THEOREM.

An LA-space B is a T_0-space relative to the natural topology on B.

Proof.The structure $\underline{\underline{B}} = \langle B, \hat{T}(B) \rangle$ is,indeed,a topological space,by 322.18. and 323.14.(Every LA-compatible sub-site of B is an open set in $\underline{\underline{B}}$.).Now B is a poset relative to its reference order (and finite, by construction);this gives the T_0-separation property. (Explicitely,for any two k_1, k_2 (distinct LA-constructions) in B,

(1°) either k_1, k_2 are incomparable relative to $\prec$,

(2°) or $k_1 \prec k_2$,

(3°) or $k_2 \prec k_1$.

In the first case,there is a B-open set,$\underline{\underline{CT}}(k_1)$ say,which does not contain k_2 (or conversely).In the latter two cases,an infimum of k_1 and k_2 exists,and is an LA-construction k say.Then $\underline{\underline{CT}}(k)$ must contain k but not the k_i with $k \neq k_i$ $(i = 1,2)$.In any case,there is some k_i in B such that $\underline{\underline{CT}}(k_i)$ contains k_i and does not contain the k_j with $i \neq j$ $(i,j = 1,2)$.But $\underline{\underline{CT}}(k_i)$ is B-open,for every k_i in B, by 322.14.So the natural topology on B is T_0.)⊠

323.18.REMARK.

LA-spaces need not be T_1 ("Fréchet spaces").Indeed,let B be an LA-space and take k_1, k_2 distinct in B such that $k_1 \prec k_2$,say. Then,for no two disjoint members B_1, B_2 in the natural topology on B one can have both

$$k_1 \in B_1, k_1 \notin B_2$$

and

$$k_2 \in B_2, k_2 \notin B_1.$$

Reason: $\underline{\underline{CT}}(k_1)$ must be a subset of $\underline{\underline{CT}}(k_2)$,the $\underline{\underline{CT}}(k_i)$'s are B-open, while,by "minimality" (i.e.,323.11. above),$\underline{\underline{CT}}(k_i)$ is contained - _qua_ set - in every B-open set having k_i as an element (i = 1,2). So,in general,an LA-space is not Fréchet.

323.19.COMMENT.(Historical.)

T_0-spaces are usually called _Kolmogorov spaces_ (cf. BOURBAKI 64) and have been - apparently - introduced first in ALEXANDROV & HOPF 35.They are not very interesting from a _geometric_ point of view say or in _real analysis_ (where one would want to start with topological spaces satisfying at least the "Hausdorff separation property").

"From a less geometrical point of view T_0-spaces can be not only interesting but also natural" (SCOTT 72) and quite a lot of work has been done around the subject during the last ten years.This was initially motivated by discoveries of Dana S.Scott,Gordon Plotkin _et alii_ in the model theory of the "type-free" lambda-calculus (SCOTT 69,72,73,75,76;PLOTKIN 72,78;BARENDREGT & LONGO 80,8X) and the semantics of programming languages (SCOTT 72a,72b; MILNE & STRACHEY 76;STOY 77;etc.).In a more general setting, T_0-topologies arise naturally in the study of continuous lattices (started in SCOTT 72c;cf. GIERZ _et al_. 80 and BANASCHEWSKI & HOFFMANN 81),in spectral theory,etc.

In particular,the Scott-topology (cf.GIERZ _et al_. 80),which is T_0, will turn out to be also involved in a mathematical semantics ("model theory") of the main AUT-languages discussed here (cf. BARENDREGT & REZUS 8X).

323.20.REMARK.

Theorem 323.17. is,in fact,an application of the well-known one-one correspondence between finite posets and finite T_0-spaces (cf.,e.g.,BIRKHOFF 48,I.11.),but the way of obtaining 323.17. here seems more natural than if one would have defined first B-closed sets and B-closure.An alternative analogy subsists between (finite) T_0-spaces (here LA-spaces) and _polyhedral complexes_ (see ALEXANDROV & HOPF 35,I.,page 132,BIRKHOFF _loc.cit_, page 14),but we leave the details to the reader.

Finally, it seems worthwhile mentioning the following fact.

323.21.DEFINITION.

Let B be an LA-space.The set

$$\mathrm{BASE}(B) = \{B_k^V : (k \text{ is in } B) \Longrightarrow (B_k^V \text{ is a reduct of } B \text{ for } k)\}$$

will be called <u>the maximal base for</u> $\hat{T}(B)$.

323.22.THEOREM.

Let B be an LA-space.Then BASE(B) is a base for the natural topology on B,viz. the maximal base for $\hat{T}(B)$.

<u>Proof</u>.By 323.11.B_k^V = CT(k),for all k in $\hat{B}$ and B is the union of finitely many members of BASE(B).☒

33. Conserving PA-correctness.

In this section we shall exploit the combinatorial analysis of correctness above in order to prove a kind of separation property for PA-compatible sites.

Accurately, if LA := CA, $Q^{-}A$, QA, it is shown that given any LA-compatible site B containing some construction k on PA, it is always possible to "extract" effectively from B a PA-compatible site B(k) such that B(k) is a subset of B and k is in B(k).

This result guarantees the intended property of "conservativity over PA" (with respect to correctness) for some class of extensions of PA.

331. "Conservation" over PA.

We need first several "structural" Lemmas establishing the "conservativity" of the correctness categories of CA, $Q^{-}A$, QA over the corresponding categories in PA.

For simplicity, we shall pay first attention to the "least rule-extension of PA" of concern here, viz. CA.

331.1. LEMMA. ($Esent_{\boxtimes}^{PA}$-conservation - in CA.)

Let B be PA-compatible and Δ be PA-admissible for B, with Eab an E-sentence on PA. If

(1) $$B,\Delta \vdash_{CA} a:b$$

then also

(2) $$B,\Delta \vdash_{PA} a:b.$$

Proof. By induction on the structure of a. Note that a cannot be τ (by 212.6.(1) above).

(1) If a is in Var, i.e., a := v, with

$$B,\Delta \vdash_{CA} v:b$$

then, by Lemma 212.9., the length of Δ is positive ($\underline{\underline{lh}}(\Delta)$ = n, n $\geqslant$ 1, say) and, for some i, $1 \leqslant i \leqslant n$,

$$\Delta := \Delta_0 [v_1 : a_1] \ldots [v_n : a_n]$$

with

$$v \equiv v_i,$$

and by, possibly, some applications of the Rules of Category Conversion (which are rules of PA) one has that b $\underline{\underline{conv}}_{\delta}$ a_i (by an argument familiar from "type-free" lambda-calculus, since $\underline{\underline{red}}_{\delta}$ is Church-Rosser; see 123.18. and 331.6.).

But Δ was supposed to be PA-admissible for B, while B was PA-compatible. So, by (Ei) which is again, a rule of PA, one has that

$$B,\Delta \vdash_{PA} v:b.$$

(2) If now $a := \underline{c}(b_1,\dots,b_q), q \geqslant 0$ (i.e., a is a head-term), each $b_j, 1 \leqslant j \leqslant q$, is a term on PA (for so was supposed to be a) and there is some construction $k_{\underline{c}}$ say on PA

(i) $$k_{\underline{c}} := \nleftarrow \Delta'; \underline{c}(v_1,\dots,v_q):b' \nrightarrow$$

(either by 212.12., if $q = 0$ or by 212.13. if $q \geqslant 1$) such that

(ii) $$\underline{c} \equiv \underline{\underline{\underline{idf}}}(k_{\underline{c}})$$

and $k_{\underline{c}}$ is sound in B, with also

(iii) $$\Delta' := \Delta_0[v_1':a_1']\dots[v_q':a_q']$$

and, whenever $q \geqslant 1$,

(iv) $$B,\Delta \vdash_{CA} b_i:a_i'[\![\bar{b} := \bar{v}]\!] \qquad (1 \leqslant i \leqslant q),$$

(v) $$B,\Delta \vdash_{CA} b \ \underline{\underline{\underline{conv}}}_\delta \ b'[\![\bar{b} := \bar{v}]\!].$$

By the inductive hypothesis one has from (iv), whenever $q \geqslant 1$, that

(vi) $$B,\Delta \vdash_{PA} b_i:a_i'[\![\bar{b} := \bar{v}]\!] \qquad (1 \leqslant i \leqslant q).$$

As B was supposed to be PA-compatible and Δ was supposed to be PA-admissible for B, we get the result from (vi), by an application of (Er-c), which is a rule of PA (if $q = 0$, this is straightforward).
This completes the induction.⊠

331.2. COROLLARY. ($Term^{PA}_{\boxtimes}$-conservation - in CA.)

Let B be PA-compatible and Δ be PA-admissible for B. If a is a term on PA and

$$B,\Delta \Vdash_{CA} a$$

then

$$B,\Delta \Vdash_{PA} a.$$

Proof. If $a \equiv \tau$, the statement is trivial (use (Ti), however). Else, use (Tr) and 331.1. above.⊠

The next proof is somewhat more instructive.

331.3. LEMMA. ($Contx^{PA}_{\boxtimes}$-conservation - in CA.)

If B is PA-compatible (and therefore a site on PA) and Δ is a context on PA such that Δ is CA-admissible for B then Δ is PA-admissible for B.

Proof. Let $\Delta := \Delta_0[v_1:a_1]\dots[v_m:a_m]$ with $\underline{\underline{lh}}(\Delta) = m, m \geqslant 0$.
First use induction on the length of Δ.

(0) If $\underline{\underline{\mathrm{lh}}}(\Delta) = 0$ then $\Delta := \Delta_0$. Then, by (Ci), which is a rule of PA, one has that Δ is PA-admissible.

(n+1) Let $\underline{\underline{\mathrm{lh}}}(\Delta) = n+1$ $(n \geqslant 0)$ and assumethe Lemma has been proved for

$$\Delta_- := \Delta_0 [v_1 : a_1] \dots [v_n : a_n].$$

That is:

$$\Delta := \Delta_- [v_{n+1} : a_{n+1}],$$

where $v_{n+1} \equiv v_m$ and $a_{n+1} \equiv a_m$.

One has that

(i)	B is PA-compatible	(hypothesis)
(ii)	Δ_- is PA-admissible for B, Δ_- on PA	(inductive hypothesis)
(iii)	a_m is a term on PA (for Δ is on PA)	(hypothesis)
(iv)	Δ is CA-admissible for B	(hypothesis)

and one has to show that

(v) Δ is PA-admissible for B.

Now, within the inductive step for $\underline{\underline{\mathrm{lh}}}(\Delta)$, use induction on the structure of a_m, taking into account (iii) above.

(n+1,1) If $a_m \equiv \tau$ then, by (Cr-1) one has, from (i), (ii) that Δ is PA-admissible for B. Indeed, v_m was supposed to be fresh for Δ_-, by (iv) above, and v_m is not in $FV(\tau) = \emptyset$.

(n+1,2) If a_m is in Var, then, by (iv), there is some $j, j < m$, such that $a_j \equiv \tau$ and $a_m \equiv v_j$, with, explicitely,

$$\Delta_- := \Delta_0 [v_1 : a_1] \dots [v_j : \tau] \dots [v_n : a_n]$$

and

$$\Delta := \Delta_- [v_m : v_j].$$

Then, by (ii), Δ_- is PA-admissible for B, so, by (Ei), which is a rule of PA,

$$B, \Delta_- \vdash_{PA} v_j : \tau.$$

Now, by (iv), v_m is fresh for Δ_- and also v_m is not in $FV(v_j)$, i.e., $v_j \not\equiv v_m$. So, by (Cr-1), which is a rule of PA, we get that Δ is PA-admissible for B.

(n+1,3) Let $a_m := \underline{c}(b_1, \dots, b_q)$, with $\underline{c} \equiv \underline{\underline{\underline{\mathrm{idf}}}}(k)$ for some construction k on PA (since a_m is a term on PA, by (iii) above; also $b_1, \dots, b_q$, if $q \geqslant 1$, should be terms on PA, by the same token). As Δ is CA-admissible for B, by (iv) above, one should have

$$B, \Delta_- \vdash_{CA} a_m : \tau$$

for $a_m \not\equiv \tau$ and therefore the degree of a_m should be 2. But then B is PA-compatible, by (i), Δ_- is PA-admissible for B, by (ii) = the inductive hypothesis, and a_m is a term on PA, by (iii). Hence by $\mathrm{Esent}_{\boxtimes}^{PA}$-conservation in CA (331.1. above), one has

$$B, \Delta_- \vdash_{PA} a_m : \tau.$$

Hence,by (Cr-2),which is a rule of PA,again,one has that Δ is PA-admissible for B. Since a_m is a PA-term,this completes the inductive steps in (n+1) and,thereby, the proof of the Lemma.⊠

331.4.COMMENT. ($Esent_{⊠}^{PA}$-,$Term_{⊠}^{PA}$- and $Contx_{⊠}^{PA}$-conservation in $Q^{-}A$,QA.)

It is now easy to see that the proofs of 331.1. through 331.3. above could have been carried through for the analogous statements with CA replaced by $Q^{-}A$ and/or QA resp.

Indeed,only the proof of 331.1. depends directly on "structural lemmas" about CA (specifically,on 212.9.,212.12. and 212.13.),but analogues of these lemmas have been seen to hold for $Q^{-}A$ and/or QA,too (cf.222. above).

This gives $Esent_{⊠}^{PA}$-conservation in $Q^{-}A$ and/or QA.

Now,$Term_{⊠}^{PA}$-conservation in $Q^{-}A$,QA follows from $Esent_{⊠}^{PA}$-conservation in $Q^{-}A$,QA on the pattern of proof of 331.2.,whereas the analogue of 331.3. in $Q^{-}A$,QA can be obtained by essentially the same kind of argument,using,of course, $Esent_{⊠}^{PA}$-conservation in $Q^{-}A$,QA at step (n+1,3).

331.5.REMARK. ($Constr_{⊠}^{PA}$-conservation in CA,$Q^{-}A$,QA.)

Let LA := CA,$Q^{-}A$,QA and assume B is a PA-compatible site.If k is a construction on PA and k is LA-sound in/for B,then k is already PA-sound in/for B. That is: we also have a kind of $Constr_{⊠}^{PA}$-conservation property in CA,$Q^{-}A$,QA. This follows easily by induction on the depth of a construction in B.

Indeed,if $\underline{depth}_B(k) = 0$ then either B = $\{k\}$ or B contains more than one element.In the first case one has to apply (Si) in order to get ∅-correctness in PA for k.In the latter case,any one of the $Site_{⊠}^{PA}$-recursion rules will do the job (according to the "structure" of k),for anyway $B_{-} = B - \{k\}$ should be PA-compatible (k is "independent" in B).

If $\underline{depth}_B(k) = n+1$,then either B is a reduced site for k,or it is not so.In the first case $B_{-} = B - \{k\}$ is PA-compatible and the result follows by $Site_{⊠}$-recursion,as earlier.Otherwise,B should contain B_k^v,so k is PA-sound in/for some subset (B_k^v) of B and the result follows by the due $Site_{⊠}^{PA}$-expansion Lemma (for sound constructions,i.e. 213.6. above,with LA := PA).

331.6.COMMENT.

Similar "conservation"-results hold,mutatis mutandis,for the "derived" correctness categories of PA.Indeed, since $\underline{red}_{\delta}$ is Church-Rosser (123.18.),one can show,by an argument familiar from lambda-calculus,that,e.g.,for all a,b on PA, a $\underline{conv}_{LA}$ b implies a $\underline{conv}_{\delta}$ b.(Note that the argument necessary here does not depend on 331.1.through 331.5.).

332. PA-separation and conservativity over PA.

Now we are able to prove our main theorem on "separating" PA-compatible sites in "rule-extensions" of PA.

We set, everywhere in the sequel, LA := PA, CA, $Q^{-}A$, QA.

First we need some trivial considerations on contexts that are PA-admissible for the empty site.

Let B be some LA-compatible site with k in B. Clearly, if $\Delta := \mathrm{ctx}(k)$ then $\mathrm{depth}_B(k)$ does not accurately characterize the "complexity" of Δ, for the latter may actually "depend on" less LA-constructions in B than k itself does.
A more sensitive "measure of complexity" for LA-admissible contexts can be found by an appropriate characterization of their analytic sets; i.e., by inspecting the depths of the elements of $\mathrm{CT}(\mathrm{ctx}(k))$.
This can be done as follows.

332.1. DEFINITION.

Let B be an LA-compatible site with k in B and $\Delta := \mathrm{ctx}(k)$.
Then the local depth (or the owndepth) of Δ in B is defined as follows:

$$\mathrm{owndepth}_B(\Delta) := \begin{cases} 0, & \text{if } \mathrm{CT}(\Delta) = \emptyset \\ \max\{\mathrm{depth}_B(k') : k' \text{ in } B\} + 1, & \text{else.} \end{cases}$$

332.2. REMARK.

If B is LA-compatible and k is in B then

$$\mathrm{owndepth}_B(\mathrm{ctx}(k)) \leqslant \mathrm{depth}_B(k) \leqslant \mathrm{depth}(B).$$

Proof. The r.h.s. of the statement is obvious from the definition of depths. To show that $\mathrm{owndepth}_B(\mathrm{ctx}(k))$ is majorized by $\mathrm{depth}_B(k)$ recall that, by definition, one has

$$\mathrm{CT}(\mathrm{ctx}(k')) \sqsubseteq \mathrm{CT}(k') \sqsubseteq B$$

for all k' in B.
The argument is by induction on sections in B. Assume $\mathrm{depth}(B) = m, m \geqslant 0$.
(0) If m = 0 then, in particular, $\mathrm{CT}(k) = \emptyset$ (for k is in B, by hypothesis) and so is $\mathrm{CT}(\mathrm{ctx}(k))$, by the inclusion noted above. Hence $\mathrm{owndepth}_B(\mathrm{ctx}(k)) = \mathrm{depth}_B(k) = 0$.
(n) Now let $m \geqslant 1$ and $\mathrm{depth}_B(k) = n$. So $1 \leqslant n \leqslant m$, by the definition of depths.
Let also, for all $n, 1 \leqslant n \leqslant m, B^{\$}_n$ be the n-section of B.
As k is supposed to be in B with depth n in B, k is in $B^{\$}_n$. By the definition of

sections we have, for all n, $1 \leq n \leq m$, $B^{\$}_{n-1} \sqsubseteq B^{\$}_{n}$.
Hence, for all n, $1 \leq n \leq m$,

$$\underline{\underline{CT}}(\underline{\underline{ctx}}(k)) \sqsubseteq \underline{\underline{CT}}(k) \sqsubseteq B^{\$}_{n-1}.$$

Now if k' is in $\underline{\underline{CT}}(\underline{\underline{ctx}}(k))$ and k' is not k then $\underline{\underline{depth}}_B(k') \leq n-1$, by definition. So we get

$$\underline{\underline{owndepth}}_B(\underline{\underline{ctx}}(k)) = \underline{\underline{max}}\{\underline{\underline{depth}}_B(k') : k' \text{ in } \underline{\underline{CT}}(\underline{\underline{ctx}}(k))\} + 1 \leq (n-1) + 1 = n.$$

But $\underline{\underline{depth}}_B(k) = n$. That is:

$$\underline{\underline{owndepth}}_B(\underline{\underline{ctx}}(k)) \leq \underline{\underline{depth}}_B(k),$$

and this completes the proof.⊠

332.3.DEFINITION.

A context in LA is PA-<u>initial</u> if it is of the form

$$\Delta := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$$

whith, whenever $\underline{\underline{lh}}(\Delta) = n, n \geq 1$, for all $j, 1 \leq j \leq n$,

(1) either $a_j \equiv \tau$

(2) or $a_j \equiv v_i$, for some $i, i < j$, and $a_i \equiv \tau$.

The following fact will be useful below.

332.4.LEMMA.

If B is LA-compatible, k is in B and $\underline{\underline{ctx}}(k)$ is a context on PA then the following (epi-)statements are equivalent:

(1) $\underline{\underline{ctx}}(k)$ is PA-admissible for the empty site.

(2) $\underline{\underline{owndepth}}_B(\underline{\underline{ctx}}(k)) = 0$.

(3) $\underline{\underline{ctx}}(k)$ is PA-initial.

<u>Proof</u>. Set $\Delta := \Delta_0[v_1:a_1]\ldots[v_m:a_m] \equiv \underline{\underline{ctx}}(k)$. Obviously, (3) implies (2).
Now (2) implies (3), by induction on the length of Δ, using the fact that k is $\emptyset$-correct (in PA).
Similarly, (3) implies (1), by induction on the length of Δ, using the due correctness rules (in order to realize that the latter are only rules of PA).
Indeed, if $\underline{\underline{lh}}(\Delta) = 0$ then Δ is PA-admissible for the empty site, by (Ci).
Moreover, if $\underline{\underline{lh}}(\Delta) = 1$ then $a_1 \equiv \tau$ and Δ is PA-admissible for the empty site, by (Cr-1).
If $\underline{\underline{lh}}(\Delta) = n+1$ then, with $\Delta\_ := \Delta_0[v_1:a_1]\ldots[v_n:a_n]$, one has $\Delta \equiv \Delta\_[v_{n+1}:a_{n+1}]$.
The hypothesis of the induction gives $\Delta\_$ PA-admissible for the empty site and, a_{n+1} is assumed to be <u>either</u> τ <u>or</u> v_i, for some $i, 1 \leq i \leq n$, where $a_i \equiv \tau$. In the first case, Δ is PA-admissible for the empty site by (Cr-1), while in the latter case one has that

$$\emptyset,\Delta_{-} \vdash_{PA} v_i : a_i \qquad (1 \leq i \leq n)$$

by (Ei),so Δ is,again,admissible for $\emptyset$,by an application of (Cr-2).
Finally,(1) implies (3),by induction on correctness in PA,i.e.,by checking the conclusions of the correctness rules in PA,producing PA-admissible contexts.⊠

332.5.COMMENT.

A deeper "context-oriented" analysis of LA-compatible sites is certainly possible,using a "nested blocks"-presentation of LA.In particular,the "nested blocks"-structure of an LA-compatible site can be described via labelled trees.The latter (called by N.G.de Bruijn,"trees of knowledge";cf. DE BRUIJN 70-02,3.7.) may be helpful in the analysis of an LA-compatible site if one wants to display a "context-sensitive organization" of the information present in that site.

However,the "nested blocks"-structure and the associated "trees of knowledge" are unessential in the language definition of an AUT-language (whether in abstract or "reference"-version) and play no actual rôle in the analysis of correctness.(Accurately,these ingredients would appear only in particular presentations of an AUT-language and there is no point in stressing their theoretical import.Cf.RUS 71.)

We can now turn back to our main task.

332.6.THEOREM.(PA-separation property).

Let B be an LA-compatible site with k in B.If k is a construction on PA then CT(k),the analytic set of k,is PA-compatible.

Proof.By induction on the depth of a construction in B.Let Δ := ctx(k).
(1) If k is in L(B,0) then $\text{depth}_B(k) = 0$.Then,by 332.2.,

$$\text{owndepth}_B(\Delta) = \text{depth}_B(k).$$

But k was supposed to be on PA,therefore so is Δ.
Hence,by Lemma 332.4.,Δ is PA-admissible for the empty site.
Now,by "induction on the structure of k",one has that k is $\emptyset$-correct in PA (by either one of the rules (Sr-1p),(Sr-2p),(Sr-1d) or (Sr-2d),which are rules of PA,too).So CT(k) = $\{k\}$ is PA-compatible.
(2) Assume k is in L(B,n+1) and suppose k depends directly on LA-constructions $k_1,\ldots,k_p$ ($p \geq 1$) and only on these,with k_i in $B_n^\$$ ($1 \leq i \leq p$).
The inductive hypothesis says that,for each k' in $B_n^\$$,CT(k') is PA-compatible.
In particular,this is true of the constructions $k_1,\ldots,k_p$ on which k depends directly.

Set, for convenience

$$B_n^{\boxtimes} = \bigsqcup_{1 \leq i \leq p} \underline{\underline{CT}}(k_i).$$

As each $\underline{\underline{CT}}(k_i)$ is PA-compatible (and a subset of B), so is $B_n^{\boxtimes}$, by 322.18.(1), (Moreover, for each i, $1 \leq i \leq p$, $\underline{\underline{CT}}(k_i) = B_{k_i}^{v}$, by 323.11.).

By the hypothesis of the Theorem B is LA-compatible. So, by Theorem 323.15., any section of B is LA-compatible and, in particular, so is $B_{n+1}^{\$}$.

By 323.18., $B_{n+1}^{\$}$ is "decomposable" into finitely many LA-compatible sites. Take the resulting "decomposition" to be "maximal".

Hence, $B_{n+1}^{\$}$ is "maximally decomposable" into LA-sites $B_{k'}^{v}$, with k' in B. In particular, B_k^{v} is LA-compatible.

The latter fact can be obtained only if

(i) k is $B_n^{\boxtimes}$-correct in LA,

and, finally, (only) if

(ii) $\Delta := ctx(k)$ is LA-admissible for $B_n^{\boxtimes}$.

Now one proceeds by "induction on the structure of k".

One has the following sub-cases.

(2.1.) for p-constructions and

(2.2.) for d-constructions.

In detail: (recall k was supposed to be on PA):

(2.1.) $k := k_{\underline{p}} \equiv \langle\!\langle \Delta;\ \underline{p}(\overline{v}) : a \rangle\!\rangle$, with $k_{\underline{p}}$ in $Pconstr^{PA}$

That is: either

(2.1.1p) $a \equiv \tau$

or

(2.1.2p) $B_n^{\boxtimes}, \Delta \vdash_{LA} a : \tau$ with a on PA.

Next

(2.2.) $k := k_{\underline{d}} \equiv \langle\!\langle \Delta;\ \underline{d}(\overline{v}) : a \rangle\!\rangle$, $\underline{\underline{where}}$ $\underline{d}(\overline{v}) ::= b$, with $k_{\underline{d}}$ in $Dconstr^{PA}$

That is: either

(2.2.1d) $a \equiv \tau$

$B_n^{\boxtimes}, \Delta \vdash_{LA} b : a \equiv \tau$ with b on PA,

or

(2.2.2d) $B_n^{\boxtimes}, \Delta \vdash_{LA} a : \tau$ with a on PA,

$B_n^{\boxtimes}, \Delta \vdash_{LA} b : a$ with b on PA.

In any case, $B_n^{\boxtimes}$ is PA-compatible and Δ is a context on PA (LA-admissible for $B_n^{\boxtimes}$), for k was supposed to be a construction on PA and $\Delta \equiv \underline{\underline{ctx}}(k)$.

This gives, by $Contx_{\boxtimes}^{PA}$-conservation (331.3. for CA; cf. 331.4. for the remaining languages) that

(iii) Δ is PA-admissible for $B_n^{\boxtimes}$.

In subcase (2.1.1p) one has,by (Sr-1p) that k is $B_n^{\boxtimes}$-correct for PA (for this is a rule of PA,too).So $\underline{\underline{CT}}(k) = B_n^{\$} \cup \{k\}$ is PA-compatible,by the same token.
In subcase (2.1.2p) one has,by $Esent_{\boxtimes}^{PA}$-conservation in LA (cf. 331.1.,331.4.) that

$$B_n^{\boxtimes}, \Delta \vdash_{PA} a : \tau$$

(for $B_n^{\boxtimes}$ is PA-compatible and Δ is PA-admissible for $B_n^{\boxtimes}$).
Hence,by (Sr-2p),one finds that k is $B_n^{\boxtimes}$-correct in PA and $\underline{\underline{CT}}(k) = B_n^{\boxtimes} \cup \{k\}$ is PA-compatible (for this is a rule of PA,too).
In subcase (2.2.1d) one has,under the same assumptions on $B_n^{\boxtimes}$ and ,that

$$B_n^{\boxtimes}, \Delta \vdash_{PA} b : a \equiv \tau$$

(by $Esent_{\boxtimes}^{PA}$-conservation in LA,again).Hence,by (Sr-1d),k is $B_n^{\boxtimes}$-correct in PA and $\underline{\underline{CT}}(k) = B_k^{\boxtimes} \cup \{k\}$ is PA-compatible.
Finally,in subcase (2.2.2d),we can replace "LA" by "PA" again,by two applications of $Esent_{\boxtimes}^{PA}$-conservation in LA,getting

$$B_n^{\boxtimes}, \Delta \vdash_{PA} b : a : \tau.$$

Hence,by (Sr-2d),which is a rule of PA,we have that k is $B_n^{\boxtimes}$-correct in PA and $\underline{\underline{CT}}(k)$ (= $B_n^{\boxtimes} \cup \{k\}$) is PA-compatible.
This completes the induction "on the structure of k" and also the main induction on depths.So the proof of the Theorem is completed.⊠

With terminology from 321.4. we have the following nice consequence from 332.6.

332.7.THEOREM.(PA-reducibility).(LA := PA,CA,$Q^{-}A$,QA).

Let B be an LA-compatible site.For any k in B,if k is on PA then B is PA-reducible (for k).

Proof.By Theorem 332.6.,$\underline{\underline{CT}}(k)$ is PA-compatible,for any k satisfying the hypothesis.Now $\underline{\underline{CT}}(k) = B_k^{v}$,by 323.11. (etc.),and $\underline{\underline{CT}}(k)$ is a site on PA (for so is k,by the hypothesis of the Theorem).That is: the reduct of B for k is a site on PA.⊠

Puttings things somewhat differently one has the following conservativity result.

332.8.THEOREM.(LA := PA,CA,$Q^{-}A$,QA).

Let B be some LA-compatible site with k in B.If k is a construction on PA then there is a PA-site B(k) such that

(1) B(k) is PA-compatible and

(2) k is in B(k).

Proof.Take,simply,B(k) := B_k^{v},by PA-reducibility (332.7.).⊠

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